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NUCLEAR FORCES AND THE STABILITY OF NEUTRON STARS

by

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ABSTRACT

The energy-momentum relationship for a neutron gas has been calculated by using Hamada-Johnston's two nucleon potential and Schwinger's variational method to determine the diagonal elements of the reaction matrix. An equation of state is constructed so that the pressure follows a simple asymtotic form at higher densities. It is known that when an amount of cold matter is assembled whose mass exceeds a certain critical mass then the system becomes unstable against collapse. We use our equation of state to integrate the general relativistic equations of hydrostatic equilibrium for the neutron gas and determine a numerical value for the critical mass.

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Chapter I. Introduction

Recently the discovery of strong celestial sources of X-rays has revived interest in the question of neutron and baryon stars. In view of this it becomes interesting to apply nuclear matter methods to an extended system consisting only of neutrons. The general problem of a many body system with realistic interactions presents great difficulties as far as exact solutions are concerned so by necessity all techniques will be approximate. Here our system is assumed to be so large that all surface effects are negligible. In such a system the existence of a translational invariance will simplify the calculations. We will assume that a nonrelativistic quantum mechanical description is valid, that the nuclear interaction between two particles may be represented by a static potential, and that the effect of many body forces is small. In the numerical work later on we use the Hamada-Johnston nuclear potential[1] which successfully accounts for the observed two nucleon data below the threshold for pion production.

Scattering experiments can only furnish information about the forces between isolated nucleons. The question remains whether these forces are the same when the particles are imbedded in a nuclear many body medium.

Gomez and co-workers[2] show that due to the exclusion principle the effect of interaction extends only to a distance small compared with the average nucleon spacing in nuclear matter, and thus during a collision two particles have only a very small probability of interacting with a third particle. The corrections to the potential energy of nuclear matter due to three body forces have been estimated by Bjorken and Wyman[3] on the basis of meson theory and are shown to be about 1 Mev per particle.

It must be remembered that the scattering of two particles of the system will differ from the scattering of free particles because a particle moving through the nuclear many body medium has its energy momentum relationship shifted by interaction with the medium. Any theory must also take into account the fact that the exclusion principle restricts intermediate state transitions to unoccupied states. Thus in spite of the fact that nuclear forces are strong and have a range comparable with the distance between nearest neighbours in a nucleus, the mean free path for nucleons incident on nuclei is large compared to the average separation of nucleons. The success of the shell model and optical model showed that in some ways an independant particle model is valid. The reaction matrix type methods which we shall use allow us to treat the particles as effectively independant provided in defining their interactions we have made allowance for the medium in which they are moving. Brueckner[4] shows that in first

approximation nuclear matter is a low density system, so it is possible to first consider the distribution of relative momentum by regarding the system as an ideal Fermi gas. This is a better approximation for the neutron gas as its density is half that of the corresponding nuclear matter problem. In particular the exclusion principle says that two particles far apart will have insufficient momentum transfer to raise them above the Fermi level so to a good approximation the particles moving in the neutron gas will behave as independent particles moving in a common potential. Other authors[5,6,7,8] have found that nuclear forces are not sufficient to bind the neutron gas at any density. We have calculated the compressibility of the neutron gas and have found it not to differ more than a factor of two or three from the compressibility of an ideal Fermi gas, so this approximation seems well founded. (See Fig.5)

The strong Fourier components required in the description of the photoelectric effect suggest a repulsive core to the two nucleon potential and direct measurements of the two nucleon correlation function at high energy show that the effect of a repulsive core is clearly present. This is usually represented by making the potential infinity positive at distances less than about 0.5×10^{-13}cm. In this case the matrix elements of $V(\vec{r})$ with respect to a determinant of plane waves are divergent so ordinary perturbation theory is not applicable. In spite of the hard core however, we know that the scattering amplitudes for nucleon-nucleon scattering remain finite quantities because the wavefunction

is zero inside the core. The radius of the hard core is much less than the average spacing of particles inside a nucleus (1.7 F where one F= 10^{-13}cm.). Moreover, the properties of the hard core are not evident except at very high energies so we expect the properties of the neutron gas to depend more strongly on the long range part of the potential and not on the detailed nature of the hard core.

In order to overcome difficulties associated with strong nuclear forces, Brueckner's theory replaces the potential $V(\vec{r})$ by an operator G which has finite matrix elements in momentum space. The operator G is defined by a non-linear integral equation which takes into account the successive interactions between any pair of particles to all orders. This integral equation is similar to the equation for the T-matrix of two body scattering theory. The G matrix is usually referred to as the Brueckner reaction matrix. The equation for G is not a scattering equation however because the total energy which is defined in terms of G itself appears in the propagator or energy denominator and not just the kinetic or free particle energy. In other words the effects of interaction which appear in the denominator change the single particle energy so that the medium becomes dispersive with the energy spectrum (which is not known apriori) being defined in terms of G itself. Numerical solutions have been obtained by iteration[11] but the requirement of self consistancy makes calculation complicated and lengthy.

Some progress in the nuclear many body problem has been made using perturbation theories and rearrangement methods. Although results seem to be approximately correct, as yet questions on the existence and convergence of the series are not clearly resolved.[12] Higher order terms in the series expansions have such complicated structures that only the first few terms may be calculated. Estimates of the errors involved in any practical calculation may be more tractable when the problem is related to a variational principle as we have done.

It should be noted in passing that the G-matrix becomes singular for attractive potentials when the momenta of two particles are nearly opposite and equal to the Fermi momentum. The singularity is a weak one and doesn't disturb computer calculation. The poles correspond to the existence. of bound states between two particles and are connected with the phenomena of superconductivity. It is possible to ignore this aspect of the problem for the purpose of calculating energy momentum relations.

The format of the thesis is as follows. Chapter II contains the formal developement of the equations to be solved for the reaction matrix and related quantities such as energy per particle. Chapter III relates the problem to a variational principle and contains the methods and results of numerical solution along with a comparison with other authors. Chapter IV is an introduction to the subject of superdense stellar configurations and a

derrivation of the relativistic equations of hydrostatic equilibrium from Einstein's field equations. Chapter V combines the results of chapters III and IV by integrating the equilibrium equations using the equation of state we have developed for the neutron gas. We discuss the question of stability of the star and give a numerical value for the critical mass beyond which there are no stable configurations of a neutron star. The question of what happens to a star at the endpoint of thermonuclear evolution which has a mass greater than the critical mass is discussed briefly.

Chapter II. The Nuclear Reaction Matrix

The total Hamiltonian for a system for N non-relativistic particles may be written;

$$H = \sum_i \left(-\frac{\hbar^2}{2m}\nabla_i^2\right) + \frac{1}{2}\sum_{i\neq j} V(\vec{r}_i,\vec{r}_j) \qquad (1.1)$$

ie. the sum of kinetic energy contributions and an interaction part involving only two body interactions (we are assuming that three body forces may be neglected). Write

$$H = H_0 + H' \qquad (1.2)$$

where H' represents the total interaction term. H_0 defines a set of wave vectors $|\Phi_i\rangle$ by

$$H_0|\Phi_i\rangle = E_i|\Phi_i\rangle \qquad (1.3)$$

If there are N particles in the system, then Φ is a determinental wave function;

$$\Phi = (N!)^{-\frac{1}{2}}\,\mathrm{Det}|\phi_1\phi_2\cdots\phi_N| \qquad (1.4)$$

where ϕ_i are single particle wave functions defined by

$$H_0\phi_i = e_i\phi_i \qquad (1.5)$$

Φ_0 is the state obtained by putting the particles in the N lowest available levels ϕ_i . H' represents the two body interaction between particles which may either change the

state of two particles or none. Singly excited states cannot be reached because of momentum conservation. In the representation defined by the set $\{\Phi\}$, the two particle matrix elements of an operator V are;

$$\langle rs|V|mn\rangle = \int \phi_r^*(1)\phi_s^*(2)V_{12}\phi_m(1)\phi_n(2)d\tau_1 d\tau_2 \quad (1.6)$$

For an infinite system the basic single particle wave functions are plane waves,

$$\phi_k = \Omega^{-\frac{1}{2}} \exp(i\vec{k}\cdot\vec{x}) \quad (1.7)$$

where $\Omega = L^3$ is the large volume containing the N particles. Periodic boundary conditions imply that each component of $\vec{k}$ can take on values $2\pi n/L$; $(n = 0, \pm1, \pm2, \cdots)$. In the limit that $\Omega\to\infty$ while N/Ω remains finite and constant, summations must be replaced by integrations. i.e.

$$\left(\frac{2\pi}{L}\right)^3 \sum_k \to \int d^3k \quad (1.8)$$

(there are $\frac{1}{8\pi^3}$ states per unit volume). In the ground state of the non-interacting system, the particles occupy a sphereof radius k_F in momentum space and

$$\frac{N}{\Omega} = \frac{k_F^3}{3\pi^2} \equiv \rho(k_F) \quad (1.9)$$

where $\rho(k_F)$ is the particle density.

The total Hamiltonian H specifies the wave vector $|\Psi_0\rangle$ and energy E of the system in the ground state.

$$H|\Psi_0\rangle = E|\Psi_0\rangle$$

Expand $|\Psi_0\rangle$ in terms of the complete set $|\Phi_n\rangle$, ie.

$$|\Psi_0\rangle = \sum_n a_n|\Phi_n\rangle \tag{1.10}$$

$$H|\Psi_0\rangle = (H_0 + H')\sum_n a_n|\Phi_n\rangle = E\sum_n a_n|\Phi_n\rangle$$

$$\sum_n a_n E_n|\Phi_n\rangle + \sum_n a_n H'|\Phi_n\rangle = E\sum_n a_n|\Phi_n\rangle$$

$$a_m E_m + \sum_n a_n\langle\Phi_m|H'|\Phi_n\rangle = Ea_m$$

$$a_m = \sum_n \frac{a_n\langle\Phi_m|H'|\Phi_n\rangle}{E-E_m}$$

$$|\Psi_0\rangle = \sum_m a_m|\Phi_m\rangle = \sum_{m\neq 0}|\Phi_m\rangle\left[\sum_n \frac{\langle\Phi_m|H'|\Phi_n\rangle}{E - E_m}\right] + a_0|\Phi_0\rangle \tag{1.11}$$

The state $|\Phi_0\rangle$ is normalized to unity but $|\Psi_0\rangle$ is normalized so that its scaler product with $|\Phi_0\rangle$ is unity. Then,

$$|\Psi_0\rangle = |\Phi_0\rangle + \sum_{m\neq 0} \frac{|\Phi_m\rangle Q\langle\Phi_m|H'|\Psi_0\rangle}{E - E_m} \tag{1.12}$$

also $$E = E_0 + \langle\Phi_0|H'|\Psi_0\rangle. \tag{1.13}$$

A projection operator Q which excludes occupied states has been inserted to take care of the exclusion principle. Assuming that the center of mass momentum of two interacting particles is zero, Q will be unity if in the intermediate states both particles are outside the Fermi sea and zero if one or both are inside. Q depends on the relative and total momentum in the intermediate states so

for non-zero center of mass momentum it will have a slightly different form. Operate on equation (1.12) by $e \equiv E - E_m$ and note that since $|\Phi_0\rangle$ is a plane wave state $e|\Phi_0\rangle = 0$. In operator form (1.12) becomes

$$e\Psi_0 = QV\Psi_0 \tag{1.14}$$

which demonstrates its similarity with a two particle Schrodinger equation (with a non-local potential).

For each pair of particles define a wave vector $|\Psi_{ij}\rangle$ which contains correlations of all pairs except (i,j) by

$$|\Psi_{ij}\rangle = |\Phi_0\rangle + \sum_{n\neq 0} \frac{|\Phi_n\rangle Q\langle\Phi_n|H' - V_{ij}|\Psi_0\rangle}{E - E_n} . \tag{1.15}$$

Define a two body operator G_{ij} by

$$G_{ij}|\Psi_{ij}\rangle = V_{ij}|\Psi_0\rangle . \tag{1.16}$$

Equations (1.12) and (1.15) combine to give

$$|\Psi_0\rangle = |\Psi_{ij}\rangle + \sum_{n\neq 0} \frac{|\Phi_n\rangle Q\langle\Phi_n|V_{ij}|\Psi_0\rangle}{E - E_n} . \tag{1.17}$$

Operate on (1.17) by V_{ij} and assume $|\Psi_{ij}\rangle \simeq |\Phi_0\rangle$ which is equivalent to neglecting many particle excitations. Then

$$\langle\Phi_m|G_{ij}|\Phi_0\rangle = \langle\Phi_m|V_{ij}|\Phi_0\rangle + \sum_{n\neq 0} \frac{\langle\Phi_m|V_{ij}|\Phi_n\rangle Q\langle\Phi_n|G_{ij}|\Phi_0\rangle}{E - E_n} . \tag{1.18}$$

We have $$E = E_0 + \langle\Phi_0|H'|\Psi_0\rangle$$

or,
$$E = E_0 + \tfrac{1}{2} \sum_{ij} \langle \Phi_0 | V_{ij} | \Psi_0 \rangle$$
$$= E_0 + \tfrac{1}{2} \sum_{ij} \langle \Phi_0 | G_{ij} | \Psi_{ij} \rangle$$
$$\simeq E_0 + \tfrac{1}{2} \sum_{ij} \langle \Phi_0 | G_{ij} | \Phi_0 \rangle \qquad (1.19)$$

Since the matrix elements of G_{ij} depend only on the single particle wave functions of particles i and j, the integral equation for G becomes,

$$\langle i'j' | G | ij \rangle = \langle i'j' | V | ij \rangle + \sum_{mn \neq ij} \frac{\langle i'j' | V | mn \rangle Q \langle mn | G | ij \rangle}{e_i + e_j - e_m - e_n} \qquad (1.20)$$

where e_i represent single particle energies defined in terms of G itself. These will be discussed in detail later.

It is conveniant to introduce the so called Møller wave matrix Ω by

$$G_{ij} = V_{ij} \Omega_{ij} \qquad (1.21)$$

Equation (1.20) in the coordinate representation for two particles in states ℓ_1 and ℓ_2 becomes

$$\langle \vec{r}_1 \vec{r}_2 | \Omega_{ij} | \ell_1 \ell_2 \rangle = \langle \vec{r}_1 \vec{r}_2 | \ell_1 \ell_2 \rangle + \sum_{\ell_1' \ell_2'} \frac{\langle \vec{r}_1 \vec{r}_2 | \ell_1' \ell_2' \rangle Q \langle \ell_1' \ell_2' | V_{ij} \Omega_{ij} | \ell_1 \ell_2 \rangle}{e_{\ell_1} + e_{\ell_2} - e_{\ell_1'} - e_{\ell_2'}}$$

let
$$\langle \vec{r}_1 \vec{r}_2 | \Omega | \ell_1 \ell_2 \rangle \equiv \Psi_{\ell_1 \ell_2}(\vec{r}_1, \vec{r}_2)$$
$$\langle \vec{r}_1 \vec{r}_2 | \Omega | \ell_1 \ell_2 \rangle \equiv \Phi_{\ell_1 \ell_2}(\vec{r}_1, \vec{r}_2) \qquad (1.23)$$

$$G(\vec{r}_1,\vec{r}_2;\vec{r}_1',\vec{r}_2') \equiv \sum_{\ell_1'\ell_2'} \frac{\Phi_{\ell_1'\ell_2'}(\vec{r}_1,\vec{r}_2)Q\Phi^*_{\ell_1'\ell_2'}(\vec{r}_1',\vec{r}_2')}{e_{\ell_1}+e_{\ell_2}-e_{\ell_1'}-e_{\ell_2'}}$$

$$\langle \vec{r}_1,\vec{r}_2|V|\vec{r}_1',\vec{r}_2'\rangle \equiv V(\vec{r}_1,\vec{r}_2;\vec{r}_1',\vec{r}_2') \tag{1.24}$$

Then,

$$\Psi_{\ell_1\ell_2}(\vec{r}_1\vec{r}_2) = \Phi_{\ell_1\ell_2}(\vec{r}_1\vec{r}_2) + \int G(\vec{r}_1\vec{r}_2;\vec{r}_1'\vec{r}_2')V(\vec{r}_1'\vec{r}_2';\vec{r}_1''\vec{r}_2'')$$

$$\times \Psi_{\ell_1\ell_2}(\vec{r}_1''\vec{r}_2'')d\vec{r}_1'd\vec{r}_2'd\vec{r}_1''d\vec{r}_2'' \tag{1.25}$$

If V is local and doesn't depend on the position of the conter of mass then,

$$\langle \vec{r}_1\vec{r}_2|V|\vec{r}_1'\vec{r}_2'\rangle = \delta(\vec{r}_1+\vec{r}_2-\vec{r}_1'-\vec{r}_2')\delta(\vec{r}_1-\vec{r}_2-\vec{r}_1'+\vec{r}_2')V(|\vec{r}_1-\vec{r}_2|) \tag{1.26}$$

$$\Psi_{\ell_1\ell_2}(\vec{r}_1\vec{r}_2) = \Phi_{\ell_1\ell_2}(\vec{r}_1\vec{r}_2) + \int G(\vec{r}_1\vec{r}_2;\vec{r}_1'\vec{r}_2')V\Psi_{\ell_1\ell_2}(\vec{r}_1'\vec{r}_2')d\vec{r}_1'd\vec{r}_2' \tag{1.27}$$

We are considering a system of infinite extent so $\Phi(r_1r_2)$ are plane waves and therefore separable in terms of center of mass and relative co-ordinates $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$. Write the quantum numbers for center of mass and relative motion corresponding to ℓ_1 and ℓ_2 as k and P. Then

$$\Psi_{kP}(\vec{r},\vec{R}) = \Phi_{kP}(\vec{r},\vec{R}) + \int G(\vec{r},\vec{R};\vec{r}',\vec{R}')V(\vec{r}')d\vec{r}'d\vec{R}'\Psi_{kP}(\vec{r}',\vec{R}') \tag{1.28}$$

The Φ's are separable, e.g.

$$\Phi_{kP}(\vec{r},\vec{R}) = \sum \Phi_k(\vec{r})\Phi_P(\vec{R}) \tag{1.29}$$

$$\text{Then,}\quad \Psi_{kP}(\vec{r}) = \Phi_k(\vec{r}) + \int G_P(\vec{r},\vec{r}')V(\vec{r}')\Psi_{kP}(\vec{r}')d\vec{r}' \qquad (1.30)$$

$$\text{and}\quad G_P(r,r') = \sum_{k'} \frac{\Phi_{k'}(\vec{r})\Phi^*_{k'}(\vec{r}')Q(\vec{P},\vec{k}')}{e_{kP} - e_{k'P}} \qquad (1.31)$$

Equation (1.28) is the standard form of the Bethe-Goldstone equation. The matrix element of G is

$$\langle\vec{k}'|G|\vec{k}\rangle = \int d\vec{r}'\Phi_{k'}(\vec{r}')V(\vec{r}')\Psi_{kP}(\vec{r}'). \qquad (1.32)$$

As is usual in the two body scattering problem, we attempt to make a partial wave expansion following the procedure of Brueckner and Gammel.[11] It is conveniant at this point to remember that we must consider the particle spin for a complete specification of the dynamical state of the system. A complication arises here because in equation (1.31) for the Green's function the summation depends on the angle between $\vec{P}$ and $\vec{k}'$ through both the energies appearing in the denominator and through the exclusion operator $Q(\vec{k}',\vec{P})$

$$\begin{aligned} Q(\vec{k}',\vec{P}) &= 0 \quad \text{for } |\vec{k}' \pm \vec{P}| \leq k_F \\ &= 1 \quad \text{for } |\vec{k}' \pm \vec{P}| > k_F \end{aligned} \qquad (1.33)$$

To remove this coupling it is usual to perform an average over angles. In particular the averaging of Q is a good approximation as the G-matrix seems to depend only weakly on the form of the exclusion principle.

$$Q(\vec{k}',\vec{P}) = \theta(k'^2+P^2+2k'P\alpha-k_F^2)\theta(k'^2+P^2-2k'P\alpha-k_F^2) \qquad (1.34)$$

where $\alpha = \cos(\vec{k}',\vec{P})$, and the θ's are step functions.

Put $a = k'^2+P^2-k_F^2$, and $b = k'P$

Define $\bar{Q}(k',P) = \frac{1}{2}\int_{-1}^{+1} Q(\vec{k}',\vec{P})d\alpha = \frac{1}{2}\int_{-1}^{+1}\theta(a-b\alpha)\theta(a+b\alpha)d\alpha \qquad (.35)$

$$\bar{Q}(k',P) = \theta(a-b)\theta(a+b) + \frac{a}{b}\theta(2a) \qquad (1.36)$$

ie.
$$\begin{aligned} \bar{Q} &= 0 \quad \text{for } k'^2+P^2 < k_F^2 \\ &= 1 \quad \text{if } |\vec{k}'-\vec{P}| > k_F \\ &= \frac{k'^2+P^2-k_F^2}{2k'P} \quad \text{otherwise} \end{aligned} \qquad (1.37)$$

The angular dependance in the denominator may be seen by expanding the single particle energies for exicited states in powers of momentum.

$$e_m + e_n = 2U_0 + U_1\cdot(k_m^2 + k_n^2) + U_2\cdot(k_m^4 + k_n^4) + \ldots \qquad (1.38)$$

$$\simeq 2U_0 + 2U_1\cdot(k'^2+P^2) + 2U_2\cdot(k'^4+2(\vec{k}'\cdot\vec{P})^2+P^4+k'^2P^2) + \ldots$$

The angle between $\vec{k}'$ and $\vec{P}$ occurs first in the quartic term. This could be eliminated by taking an angle average of the term $(\vec{k}'\cdot\vec{P})^2$. Bethe[14] has argued to neglect the single particle potentials in the excited states for a pure neutron gas. We follow his example and proceed with the partial wave expansion. The non-interacting two particle wave fucntion for relative motion may be expanded as

$$\Phi_k(\vec{r}) = \exp(i\vec{k}\cdot\vec{r})\chi_s^{m_s} = \sum_{\ell=0}^{\infty} c_\ell j_\ell(kr)\, Y_\ell^0(\hat{R},\hat{r})\chi_s^{m_s} \qquad (1.39)$$

where $$c_\ell = (2\ell+1)i^\ell\left(\frac{4}{2\ell+1}\right)^{\frac{1}{2}}$$

and $f(\hat{R},\hat{r})$ means a function of the angles between $\vec{k}$ and $\vec{r}$. In the presence of a tensor force, the orbital angular momentum number ℓ is not a constant of motion but the total angular momentum is. Equation (1.39) should be expanded in terms of eigenfunctions of J, that is $F_L^{Jm_Js}$. These functions are related to the above by,

$$F_L^{Jm_Js} = \sum_{m_s=-1}^{+1} Y_L^{J-m_s}\, \chi_s^{m_s}\, C(J,m_J;L,m_J-m_s;s,m_s). \qquad (1.40)$$

The C's in the above expression are Clebsch-Gordan coefficients. Inverting equation (1.40),

$$Y_L^{m_L}\chi_s^{m_s} = \sum_{J=L-1}^{L+1} F_L^{J,m_L+m_s,s}\, C(J,m_L+m_s;L,m_L;sm_s). \qquad (1.41)$$

Equation (1.39) then becomes,

$$\Phi_k(\vec{r}) = \sum_{J=0}^{\infty}\ \sum_{\ell=J-1}^{J+1} c_\ell\, j_\ell(kr)\, F_\ell^{Jm_ss}(\hat{R},\hat{r})\, C(Jm_s;\ell 0;sm_s) \qquad (1.42)$$

We want a similar expansion for $\Psi_k(r)$. It must be remembered here that the tensor force couples the states

for $s = 1$, $\ell = J \pm 1$, (the state $s = 1$, $\ell = J$ is not coupled with any other). We adopt a notation for distinguishing the states $\ell = J \pm 1$ by putting subscripts on the partial wave functions U.

No tensor force	With tensor force

$$j_{J\pm1}(kr)\, F_{J-1}^{Jm_J s}(\hat{r},\hat{r}) \quad \rightarrow \quad U_{J\pm1,J-1}(kr) F_{J-1}^{Jm_J s} + U_{J\pm1,J+1} F_{J+1}^{Jm_J s} \tag{1.43}$$

Denote the uncoupled triplet states by

$$j_J(kr)\, F_J^{Jm_J s} \quad \rightarrow \quad U_{JJ}^{Js}(kr)\, F_J^{Jm_J s}.$$

Then the expansion for $\Psi_k(r)$ is (dropping the index P on Ψ),

$$\Psi_k(\vec{r}) = \sum_{J=0}^{\infty} \sum_{\ell=J-1}^{J+1} \sum_{\ell'=J-1}^{J+1} {}_{\ell}U_{\ell\ell'}^{Js}(r)\, F_{\ell'}^{Jm_S S}(\hat{r},\hat{r})\, C(Jm_S;\ell 0;sm_S). \tag{1.44}$$

A similar expansion for the Green's function is,

$$G(\vec{r},\vec{r}') = \sum_{\ell=0}^{\infty} c_\ell i^\ell\, G_\ell(r,r')\, Y_\ell^0(\hat{r},\hat{r}') \tag{1.45}$$

where
$$G_\ell(r,r') = \frac{1}{2\pi^2} \int_0^\infty \frac{k'^2 dk' j_\ell(k'r)\, j_\ell(k'r')\, \bar{Q}(k';P)}{e(k,k';P)}. \tag{1.46}$$

Insert these expressions into equation (1.28) and drop the sum over J which is common to all terms.

$$\sum_{\ell\ell'} c_\ell \, U^{Js}_{\ell\ell'}(r) \, F^{Jm_S s}_{\ell'}(\hat{R},\hat{r}) \, C(Jm_S;\ell 0;sm_S) =$$

$$\sum_{\ell} c_\ell \, j_\ell(kr) \, F^{Jm_S s}_{\ell}(\hat{R},\hat{r}) \, C(Jm_S;\ell 0;sm_S)$$

$$+ \sum_{\ell} \int d\vec{r}' c_\ell (-i)^\ell G_\ell(r,r') \, Y^0_\ell(\hat{r},\hat{r}') \, V(r')$$

$$\times \sum_{\ell'\ell''} c_{\ell'} U^{Js}_{\ell'\ell''}(r') \, F^{Jm_S s}_{\ell''}(\hat{R},\hat{r}') \, C(Jm_S;\ell' 0;sm_S) \qquad (1.47)$$

Integrate over the angles of r'

$$(-i)^\ell c_\ell \int dr' \, Y^0_\ell(\hat{r},\hat{r}') \, V(r') \, F^{Jm_S s}_{\ell''}(\hat{R},\hat{r}') =$$

$$4\pi \, F^{Jm_S s}_{\ell}(\hat{R},\hat{r}) \, V^{Js}_{\ell\ell''}(r') \qquad (1.48)$$

where $V_{\ell\ell''}(r) = \int d\hat{r} \, F^{Jm_S s}_{\ell}(\hat{R},\hat{r}) \, V(r) \, F^{Jm_S s}_{\ell''}(\hat{R},\hat{r})$ (1.48)

and we have used the relationship

$$Y^0_\ell(\hat{r},\hat{r}') = \left(\frac{4}{2\ell+1}\right)^{\frac{1}{2}} \sum_{m=-1}^{1} Y^m_\ell(\hat{R},\hat{r}') \, Y^m_\ell(\hat{R},\hat{r}) \qquad (1.50)$$

We may write $V^{Js}_{\ell\ell''}(r) = S^{Js}_{\ell\ell''} \, v_T(r)$ for tensor forces

$= \delta_{\ell\ell''} \, v_c(r)$ for central forces.

Here $v_T(r)$ and $v_c(r)$ are radial dependances and the S's are numbers listed by Ashkin and Wu[15]. Put equation (1.48) into (1.47), take the scaler product with $F^{Jm_s s}_{L'}(\hat{R},\hat{r})$, integrate over angles of $\vec{r}$ (change the dummy index ℓ to ℓ' on the left hand side). Then we get,

$$\sum_{\ell'} c_{\ell'} U^{Js}_{\ell' L'}(r)\, C(Jm_s;\ell' 0;sm_s) = c_{L'} j_{L'}(kr)\, C(Jm_s;L'0;sm_s)$$

$$+ 4\pi \sum_{\ell'\ell''} \int r'^2 dr'\, G_{L'}(r,r')\, V^{Js}_{L'\ell''}(r)\, c_{\ell'}\, U^{Js}_{\ell'\ell''}(r')\, C(Jm_s;\ell'' 0;sm_s) \tag{1.51}$$

Now multiply through by $C(Jm_s;L0;sm_s)$ and sum over m_s using the orthogonality relationship

$$\sum_{m_s=-1}^{1} C(Jm_s;L0;sm_s)\, C(Jm_s;\ell' 0;sm_s) = \delta_{L\ell'}\left(\frac{2J+1}{2\ell'+1}\right) \tag{1.52}$$

which gives (changing $L\to\ell$ and $L'\to\ell'$)

$$U^{Js}_{\ell\ell'} = j_\ell(kr)\delta_{\ell\ell} + 4\pi \sum_{\ell''} \int r'^2 dr' G_{\ell'}(r,r') V_{\ell\ell''}(r') U_{\ell\ell''}(r') \tag{1.53}$$

The equations for all angular momentum states have similar form so let us suppress indices for a moment.

$$U(r) = U_0(r) + 4\pi \int_0^\infty G(r,r')\, V(r')\, U(r')\, r'^2 dr' \tag{1.54}$$

This must be modified for potentials with hard cores because $V(r) = \infty$ over the core region. Brueckner and Gammel[11] show for the two body scattering problem the replacement

$$V(r)U(r) = \lambda\delta(r-c) \tag{1.55}$$

for $r \leq c$ leads to the correct Schrodinger equation for two body scattering. λ is determined by the boundary condition $U(c) = 0$.

$$U(r) = U_0(r) + 4\pi c^2 \lambda G(r,c) + 4\pi \int_c^\infty G(r,r')U(r')V(r')r'^2 dr' \tag{1.56}$$

$$U(c)=0 \text{ implies } \lambda = \frac{-\int_c^\infty G(c,r')V(r')U(r')r'^2 dr' + U_0(r)}{G(c,c)} \tag{1.57}$$

Equation (1.54) is replaced by

$$U(r) = f(r) + 4\pi \int_c^\infty K(r,r')V(r')U(r')r'^2 dr' \tag{1.58}$$

with $$K(r,r') = G(r,r') - \frac{G(r,c)\,G(c,r')}{G(c,c)}$$

and $$f(r) = U_0(r) - U_0(c)\,\frac{G(c,r)}{G(c,c)}.$$

For the many body system, the situation is not as simple. However, Bethe and Goldstone[22] show that replacing equation

(1.54) by (1.58) and setting $U(r) = 0$ for $r < c$ leads to negligible error. Brueckner and Gammel[11] have also considered this problem for a hard core potential with no attraction, taking S-states only. They write

$$V(r)U(r) = \lambda\delta(r-c) + f(r) \quad \text{for } r < c$$

and then show the effects of the correction term are about 0.1% for nuclear matter at equilibrium density. We therefore feel justified in following the same approach for a neutron system. The general treatment taking into account angular momentum states is exactly the same as was just done and consists of making the following replacements in Equation (1.53).

$$j_\ell(kr) \to f_\ell(kr) \equiv j_\ell(kr) - j_\ell(kc)\frac{G_\ell(c,r)}{G_\ell(c,c)} \tag{1.60}$$

$$G_\ell(r,r') \to K_\ell(r,r') \equiv G_\ell(r,r') - \frac{G_\ell(r,c)\,G_\ell(c,r')}{G_\ell(c,c)} \tag{1.61}$$

Inserting equations (1.42) and (1.44) into (1.32), we find for the diagonal elements of the reaction matrix,

$$\langle k|G|k\rangle = 4\pi\sum_J \sum_{\ell=J-1}^{J+1}(2J+1)\int_0^\infty r^2 dr\, j_\ell(kr)\left[\sum_{\ell'=J-1}^{J+1} V^{Js}_{\ell\ell'}U^{Js}_{\ell\ell'}\right] \tag{1.62}$$

This expression may be separated into two contributions; one representing the core volume term, the other the effect of the core surface and outer potential. The procedure is the same as before. Replace $V(r')U(r')$ by $\lambda\delta(r'-c)$ for $r' \leqq c$, where λ is given by (1.57) and the core radius is assumed the same in all states. Then,

$$\langle k|G|k\rangle \equiv \langle k|G|k\rangle_{out} + \langle k|G|k\rangle_{core}$$

$$= 4\pi \sum_J (2J+1) \sum_\ell \left[\frac{-j_\ell(kc)}{G_\ell(c,c)} + \int_c^\infty r^2 dr\, f_\ell(kr) \sum_{\ell'=J-1}^{J+1} V^{Js}_{\ell\ell}(r)\, U^{Js}_{\ell\ell}(r) \right] \quad (1.63)$$

$$+ \langle k|G|k\rangle_{core}.$$

The contribution from the core was calculated from equation (2.3) of reference 16.

Chapter III. The Equation of State for a Neutron Gas

In attempting a particular solution of the problem for an extended system consisting only of neutrons, we must remember that the exclusion principle states that the wave function must be antisymetric under exchange of all particle co-ordinates. For the neutron gas we will have only to consider singlet states of even parity and triplet states of odd parity. We believe that the system will be adequately described by considering only the partial waves for $\ell = 0$ and 1. i.e. the contibutions to the reaction matrix for $\ell > 1$ are very small.[16] We make the further assumption here that the coupling between the $^3P_2$ and $^3F_2$ states is sufficiently weak that we may consider the tensor force for the $^3F_2$ state to be identically zero.

Equation (1.53) can be solved only by making certain approximations whose validity is difficult to assess. We now attempt an approximate solution by using a variational technique similar to Schwinger's.[17] Consider equation (1.53) in the form

$$|U_\ell\rangle = |f_\ell\rangle + K_\ell(r,r')\, V(r')|U_\ell\rangle. \qquad (2.2)$$

The reaction amplitude for the outer potential is

$$g_\ell = \langle f_\ell|V|U_\ell\rangle. \qquad (2.3)$$

From equation (2.2) we may write

$$\frac{|\langle f_\ell|V|U_\ell\rangle|^2}{\langle f_\ell|V|U_\ell\rangle} = \langle U_\ell|V|U_\ell\rangle - \langle U_\ell|VK_\ell V|U_\ell\rangle$$

or

$$\langle f_\ell|V|U_\ell\rangle = \frac{|\langle f_\ell|V|U_\ell\rangle|^2}{\langle U_\ell|V|U_\ell\rangle - \langle U_\ell|VK_\ell V|U_\ell\rangle} . \qquad (2.4)$$

Consider the quantity

$$J(U_{\ell T}) = \frac{|\langle f_\ell|V|U_{\ell T}\rangle|^2}{\langle U_{\ell T}|V|U_{\ell T}\rangle - \langle U_{\ell T}|VK_\ell V|U_{\ell T}\rangle} \qquad (2.5)$$

where $|U_{\ell T}\rangle$ is some trial wave function satisfying the same boundary conditions as $|U_\ell\rangle$, namely $U_\ell(r\to\infty) \to j_\ell(kr)$ and $U_\ell(c) = 0$ where c is the radius of the hard core. The quantity $J(U_{\ell T})$ has the property of being homogeneous in $|U_{\ell T}\rangle$ so if we multiply $|U_{\ell T}\rangle$ by some arbitrary constant factor, the expression remains unchanged. Note also that $|U_{\ell T}\rangle$ is important only if it is within the range of nuclear forces. Also, the error in $J(U_{\ell T})$ is proportional to the square of the error in $|U_{\ell T}\rangle$. Write

$$J(U_{\ell T}) = \frac{P^2}{Q}; \text{ where } \quad P = \langle f_\ell|V|U_{\ell T}\rangle \qquad (2.6)$$

$$Q = \langle U_{\ell T}|V|U_{\ell T}\rangle - \langle U_{\ell T}|VKV|U_{\ell T}\rangle$$

Consider the variation of $J(U_{\ell T})$ with respect to arbitrary variations in $|U_{\ell T}\rangle$ subject to the condition that $|U_{\ell T}\rangle$ satisfy the same boundary conditions as $|U_\ell\rangle$.

$$J = \frac{2P}{Q}.\delta P - \frac{P^2}{Q^2}\delta Q \quad \text{where} \quad \delta P = \langle f_\ell|V|\delta U_{\ell T}\rangle$$

$$\delta Q = 2\langle U_{\ell T}|V-VKV|\delta U_{\ell T}\rangle$$

$$\delta J = \frac{2P}{Q^2}\left[\langle f_\ell|V|\delta U_{\ell T}\rangle\langle U_{\ell T}|V-VKV|U_{\ell T}\rangle - \langle f_\ell|V|U_{\ell T}\rangle\langle U_{\ell T}|V-VKV|\delta U_{\ell T}\rangle\right.$$

$$= \frac{2P}{Q^2}\left[\langle U_{\ell T}|V-VKV|U_{\ell T}\rangle\langle f_\ell|V-\langle f_\ell|V|U_{\ell T}\rangle\langle U_{\ell T}|V-VKV|\right]|\delta U_{\ell T}\rangle$$

$$= \frac{2P}{Q^2}\{\langle F_\ell|\delta U_{\ell T}\rangle\} \qquad (2.7)$$

$$\text{where } \langle F_\ell| = \langle f_\ell|V|U_{\ell T}\rangle\left[\frac{\langle U_{\ell T}|V-VKV|U_{\ell T}\rangle}{\langle f_\ell|V|U_{\ell T}\rangle} \cdot \langle f_\ell|V-\langle U_{\ell T}|(V-VKV)\right] \qquad (2.8)$$

For $\delta J = 0$ it is necessary that $\langle F_\ell| = 0$, i.e. necessary that $|U_{\ell T}\rangle$ satisfy the integral equation (2.2). Note that if $|U_{\ell T}\rangle = |U_\ell\rangle$, then

$$J(U_{\ell T}) = \langle f_\ell|V|U_\ell\rangle \qquad (2.9)$$

which is the contribution of the outer potential to the reaction matrix. It is hoped that if the trial wave function is well choosen, then varying some arbitrary

parameter(s) to find a stationary value of $J(U_{\ell T})$ will give a wave function that reasonably resembles the ground state wave function of the system. We have choosen a variational approach to this problem because it will be possible to put a bound on the ground state energy of the system according to the general statements that can be made of variational principles. The following considers this problem. J_a is the quadratic functional defined by a potential V_a and arbitrary $|U_{\ell T}>$,(satisfying the boundary conditions of course).

$$J_a = \frac{|<f_\ell|V_a|U_{\ell T}>|^2}{<U_{\ell T}|V_a|U_{\ell T}> - <U_{\ell T}|V_a K V_a|U_{\ell T}>} \qquad (2.10)$$

Consider another potential $V_b = V_a + \varepsilon$ (2.11)

where ε is infinitesimal, real, and positive. Define J_b by

$$J_b = \frac{|<f_\ell|V_b|U_{\ell T}>|^2}{<U_{\ell T}|V_b|U_{\ell T}> - <U_{\ell T}|V_b K V_b|U_{\ell T}>}. \qquad (2.12)$$

$$J_b = \frac{|<f_\ell|V_a|U_{\ell T}>|^2 + \varepsilon\{<f_\ell|V_a|U_{\ell T}><U_{\ell T}|f_\ell> + O(\varepsilon^2) + \ldots}{<U_{\ell T}|V_a|U_{\ell T}> - <U_{\ell T}|VKV|U_{\ell T}> + 2\varepsilon<U_{\ell T}|U_{\ell T}> - 2\varepsilon U_{\ell T}|V_a K|U_{\ell T}>}$$

Write $J_a = \frac{A}{B}$, and $J_b = \frac{A + \varepsilon C}{B + \varepsilon D}$ (2.13)

where $A = |\langle f_\ell | V_a | U_{\ell T} \rangle|^2$

$$B = \langle U_{\ell T} | V_a | U_{\ell T} \rangle - \langle U_{\ell T} | V_a K V_a | U_{\ell T} \rangle$$

$$C = 2\langle f_\ell | V_a | U_{\ell T} \rangle \langle U_{\ell T} | f_\ell \rangle \qquad (2.13)$$

$$D = \langle U_{\ell T} | U_{\ell T} \rangle - 2\langle U_{\ell T} | V_a K | U_{\ell T} \rangle$$

Expand $J_b = J_b(\varepsilon=0) + \varepsilon J_b'(\varepsilon=0) + O(\varepsilon^2) + \ldots$

$$J_b'(\varepsilon=0) = \frac{BC - DA}{B} \quad ; \quad J_b(\varepsilon=0) = J_a$$

Therefore $$(J_b - J_a) = \frac{\varepsilon}{B^2}\left[BC - DA\right] \qquad (2.14)$$

Assume that V_a is an attractive potential, ie.

$$V_a | U_{\ell T} \rangle = -W_0 | U_{\ell T} \rangle \; ; \quad W_0 \geqq 0$$

Consider BC - DA,

$$= \{-W_0 \langle U_{\ell T} | U_{\ell T} \rangle - W_0^2 \langle U_{\ell T} | K | U_{\ell T} \rangle\} \; (-2W_0) |\langle f_\ell | U_{\ell T} \rangle|^2$$

$$-W_0^2 |\langle f_\ell | U_{\ell T} \rangle|^2 \{\langle U_{\ell T} | U_{\ell T} \rangle + 2W_0 \langle U_{\ell T} | K | U_{\ell T} \rangle\}$$

$$= W_0^2 \langle U_{\ell T} | U_{\ell T} \rangle |\langle f_\ell | U_{\ell T} \rangle|^2 \qquad (2.15)$$

Therefore, $$J_b - J_a = \frac{\varepsilon W_0^2}{B^2} \langle U_{\ell T} | U_{\ell T} \rangle |\langle f_\ell | U_{\ell T} \rangle|^2 \geqq 0 \qquad (2.16)$$

The requirement that ε be infinitesimal can be dropped as any finite difference in potential may be constructed from a number of infinitesimal ones with the corresponding shifts in J. We may now write the following theorem.

1. For attractive potentials

   If $\langle\phi|V_b|\phi\rangle \geqq \langle\phi|V_a|\phi\rangle$ for all ϕ, then $J_b \geqq J_a$.

2. For repulsive potentials

   If $\langle\phi|V_b|\phi\rangle \leqq \langle\phi|V_a|\phi\rangle$ for all ϕ, then $J_b \leqq J_a$.

It will be conveniant now to consider a different normalization of the wave function. Define

$$K'(r,r') \equiv K(r,r')\, V(r'), \text{ and } \quad \Omega \equiv V(1 - K')$$

where Ω is a real Hermitian operator. Then

$$J = \frac{|\langle f_\ell|V|U_{\ell T}\rangle|^2}{\langle U_{\ell T}|\Omega|U_{\ell T}\rangle}.$$

Assume $\langle f_\ell|V|U_{\ell T}\rangle = 1$, then

$$J(U_{\ell T}) = \frac{1}{\langle U_{\ell T}|\Omega|U_{\ell T}\rangle}. \qquad (2.17)$$

For an attractive potential V(r), Schwartz's Inequality states for any ϕ and $U_{\ell T}$;

$$\langle\phi|V|\phi\rangle \leq \frac{\langle\phi|V|U_{\ell T}\rangle\langle U_{\ell T}|V|\phi\rangle}{\langle U_{\ell T}|V|U_{\ell T}\rangle} \equiv \langle\phi|V_s|\phi\rangle, \qquad (2.18)$$

Where V_s is a non-local operator. Consider $|U^{V_s}\rangle$ the wave function defined by the non-local potential and

$$|U^{V_s}\rangle = |f_\ell\rangle + KV_s|U^{V_s}\rangle. \qquad (2.19)$$

Then $\langle U_{\ell T}|V|U^{V_s}\rangle = \langle U_{\ell T}|V|f_\ell\rangle + \langle U_{\ell T}|VKV_s|U^{V_s}\rangle$, or

$$\langle U_{\ell T}|V|U^{V_s}\rangle = 1 + \frac{\langle U_{\ell T}|VKV|U_{\ell T}\rangle\langle U_{\ell T}|V|U^{V_s}\rangle}{\langle U_{\ell T}|V|U_{\ell T}\rangle}.$$

$$\langle U_{\ell T}|V|U_{\ell T}\rangle - \langle U_{\ell T}|VK'|U_{\ell T}\rangle = \frac{\langle U_{\ell T}|V|U_{\ell T}\rangle}{\langle U_{\ell T}|V|U^{V_s}\rangle}$$

$$= \langle U_{\ell T}|\Omega|U_{\ell T}\rangle \equiv \frac{1}{J(U_{\ell T})} \qquad (2.20)$$

The reaction amplitude for the non-local potential V_s is given by

$$\langle f_\ell|V_s|U^{V_s}\rangle \equiv \frac{\langle f_\ell|V|U_{\ell T}\rangle\langle U_{\ell T}|V|U^{V_s}\rangle}{\langle U_{\ell T}|V|U_{\ell T}\rangle} = J(U_{\ell T}) \text{ from 2.20.}$$

Any $|U_{\ell T}>$ that we choose will be equivalent to picking some V_s satisfying (2.18). $J(U_{\ell T})$ will be the reaction amplitude for the potential V_s, and from the previous theorem (for attractive potentials),

$$J(U_\ell) \leqq J(U_{\ell T}) \tag{2.21}$$

For an attractive potential V, the quantity $J(U_{\ell T})$ provides an upper bound for the reaction matrix due to the potential V. Similarly for a repulsive potential the quantity $J(U_{\ell T})$ provides a lower bound.

For any practical solution we are still faced with the fact that the energy denominator appearing in the Green's function $G_\ell(r,r')$ depends on the reaction matrix itself. It contains both the kinetic and potential energies of the initial and intermediate states. (see equation 1.20)

$$e(\vec{k},\vec{k}',\vec{P}) = e_i + e_j - e_m - e_n$$

$$= e(\tfrac{\vec{P}}{2} + \vec{k}) + e(\tfrac{\vec{P}}{2} - \vec{k}) - e(\tfrac{\vec{P}}{2} + \vec{k}') - e(\tfrac{\vec{P}}{2} - \vec{k}')$$

Here e_i are single particle energies,

$$e_i = \frac{k_i^2}{2m} + U(k_i)$$

and $U(k_i)$ are single particle potential energies seen by a particle of momentum k_i.

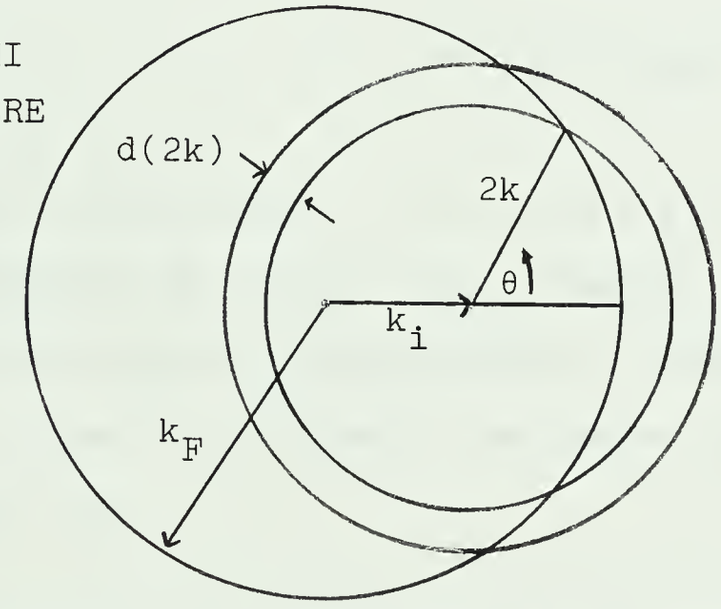
For the pure neutron gas, Bethe[14] has argued to neglect the single particle potential energy in the excited states. Then

$$e(k,k',P) = k^2-k'^2 + U(\vec{\frac{P}{}} +\vec{k}) + U(\vec{\frac{P}{}} -\vec{k}). \quad (2.23)$$

For particles inside the Fermi sea,

$$U(k_i) = \frac{1}{\Omega} \int_0^\infty \langle k|G|k\rangle \, N(k_i,k)dk, \quad (2.24)$$

where $N(k_i,k)dk$ is the number of particles in a degenerate Fermi gas of N particles which have a relative momentum between k and k+dk with respect to a given particle of fixed momentum k_i. This function may be obtained as follows.



For a given k_i, the maximum of k is $\frac{1}{2}(k+k_i)$. The factor of two appears because the relative momentum is μv where v is the relative velocity and μ is the reduced mass = m/2. For $k < \frac{1}{2}(k_F-k_i)$, $\vec{k}$ can go over all directions. Then $N(k_i,k)dk$ is the number of particles in a spherical shell with radius 2k and thickness d(2k).

For this case,

$$N(k_i,k)dk = \frac{N\ 4\pi(2k)^2 d(2k)}{(4\pi/3)\ k_F^3} = N\ \frac{24k^2}{k_F^3}\ dk. \qquad (2.25)$$

For $k > \frac{1}{2}(k_F - k_i)$, θ cannot go into the solid angle $2\pi(1-\cos\theta)$ From the diagram,

$$\cos(\theta) = \frac{k_F^2 - k_i^2 - 4k^2}{4k \cdot k_i}.$$

Then,
$$N(k_i,k)dk = \frac{N(2k)^2 d(2k)}{(4\pi/3)\ k_F^3}\left[\ 4\pi - 2\pi(1-\cos\theta)\ \right]$$

$$= N \cdot 12(1+\cos\theta)\ \frac{k^2}{k_F^3} \cdot dk \qquad (2.26)$$

The total energy of the ground state of the system may be expressed in terms of the diagonal elements of the reaction matrix by taking matrix elements of the total Hamiltonian (1.1) between the states $|\Phi_0\rangle$ and $|\Psi_0\rangle$ with $\langle\Phi_0|\Psi_0\rangle = 1$.

$$E = \langle\Phi_0|H|\Psi_0\rangle = \sum_i \langle\Phi_0|T_i|\Psi_0\rangle + \tfrac{1}{2}\sum_{ij}\langle\Phi_0|V_{ij}|\Psi_0\rangle$$

$$= \sum_i \frac{k_i^2}{2} + \tfrac{1}{2}\sum_{ij}\langle ij|G|ij\rangle \qquad (2.27)$$

The average of the kinetic energy term over all occupied levels of the degenerate Fermi gas is $\overline{T} = \frac{3}{5}\frac{\hbar^2}{2m}k_F^2$. The average potential energy of the particles in the ground state may be written in terms of the single particle potentials or by finding the average value of G when k_i and k_j take on all values in the Fermi sea.

$$\overline{U} = \frac{\int_{k_i<k_F} U(k_i)\, d^3k_i}{\int_{k_i<k_F} d^3k_i} = \frac{1}{\Omega}\int_0^{k_F} P(k,k_F)\, \langle k|G|k\rangle\, dk \qquad (2.28)$$

Here $P(k,k_F)dk$ is the number of pairs in a degenerate Fermi gas of N identical particles with relative momentum between k and k+dk. This function may be obtained from $N(k_i,k)dk$ by integrating k_i over all possible values (and normalizing).

$$P(k,k_F)dk = \int_0^{k_F} \frac{N(k_i,k)dk\cdot 4\pi k_i^2 dk_i}{(4\pi/3)\,k_F^3} = \frac{24Nk^2dk}{k_F^3}\left\{1-\frac{3k}{2k_F}+\frac{1}{2}\left(\frac{k}{k_F}\right)^3\right\} \qquad (2.29)$$

For low enough densities it is known that the diagonal elements of the reaction matrix have approximately a simple quadratic behavior for $k < k_F$.[16] If this is the case we find from equations (2.28) and (2.29) that $\overline{U}$ may be evaluated by finding the diagonal elements of G at an average relative momentum $\overline{k} = \sqrt{.3}\,k_F$. Then $\overline{U} = \rho\langle\overline{k}|G|\overline{k}\rangle$ and the average energy per particle is

$$E = \frac{3}{5}\frac{\hbar^2}{2m} k_F^2 + \tfrac{1}{2}\rho(k_F) \langle \bar{k}|G|\bar{k}\rangle \qquad (2.30)$$

A similar treatment for the single particle potential is possible. Assume that we may replace the integration in (2.24) by the reaction matrix evaluated at an average relative momentum defined by $\underline{k}^2 = \tfrac{1}{4}(k_i^2 + 0.6\ k_F^2)$. Then $U(k_i)$ is approximately quadratic so we may write the energy denominator as;

$$e(k,k',P) \simeq k^2 - k'^2 + 2U\left[(P^2 + k^2)^{\frac{1}{2}}\right]$$

$$\simeq k^2 - k'^2 + 2\rho(k_F)\ G\left(\tfrac{1}{2}\sqrt{P^2+k^2+0.6k_F^2},\ \tfrac{1}{2}\sqrt{P^2+k^2+0.6k_F^2}\right)$$

Most authors treat the effect of the center of mass momentum by replacing P by its average over the Fermi sea,

$$P^2 = \frac{3}{5} k_F(k_F - k)\left\{1 + \frac{k^2}{3k_F(2k_F+k)}\right\}: \quad k < k_F$$

$$= 0 \quad : \quad k > k_F. \qquad (2.31)$$

We have used the value $P = \sqrt{.3}\ k_F$ which corresponds to the peak in the distribution of the number of pairs of particles with center of mass momentum P. Since we need only to evaluate the matrix elements $\langle \bar{k}|G|\bar{k}\rangle$, (with $\bar{k} = \sqrt{.3}\ k_F$) to find the average energy per particle, the energy denominator is;

$$e(k,k',P) = \bar{k}^2 - k'^2 + 2\rho(k_F)\langle \bar{k}|G|\bar{k}\rangle . \qquad (2.32)$$

It is known[19] that if we replace the energy denominator by a free particle propagator and put $\bar{Q} = 1$ for all states, then the diagonal elements of the reaction matrix may be written in terms of the phase shifts.

$$\langle k|G|k\rangle = \frac{-4\pi}{k} \sum_J (2J+1)\, A_\ell\, \delta_\ell^{J,S}$$

where $A_\ell = 1/4$ for even ℓ and $3/4$ for odd ℓ. This is not a particularly good approximation for nuclear matter as it doesn't account for the observed saturation[11], but for the neutron gas at low densities it is close to the values of the reaction matrix. For the first numerical iteration of the problem we use the phase shift approximation for G in the denominator, including only S-wave contributions.

Let us write the energy denominator as

$$e(k,k',P) = -\{\gamma^2(k) + k'^2\} \qquad (2.34)$$

If we put the center of mass momentum identically equal to zero and neglect the exclusion principle, then the Green's function may be evaluated in closed form.

$$G_\ell(r,r') = \frac{-2}{\pi} \int_0^\infty \frac{j_\ell(k'r)\, j_\ell(k'r')\, dk'}{(k'^2+\gamma^2)}$$

$$= \gamma j_\ell(i\gamma r)\, h_\ell^{(1)}(i\gamma r') \;;\quad \text{for } r < r'$$

$$= \gamma j_\ell(i\gamma r')\, h_\ell^{(1)}(i\gamma r) \;;\quad \text{for } r > r'. \tag{2.35}$$

Then,

$$U_\ell(kr) = j_\ell(kr) + \gamma \int_0^r j_\ell(i\gamma' r)\, h_\ell^{(1)}(i\gamma r) V(r') U_\ell(kr') r'^2 dr'$$

$$+ \gamma \int_r^\infty j_\ell(i\gamma r)\, h_\ell^{(1)}(i\gamma r')\, V(r')\, U_\ell(kr')\, r'^2 dr'.$$

Put $$U_\ell = j_\ell(kr) + A\gamma\, h_\ell^{(1)}(i\gamma r) + B\gamma j_\ell(i\gamma r) \tag{2.36}$$

The boundary conditions are; 1. $U_\ell(kc) = 0$

2. $U_\ell(\infty) = j_\ell(kr)$.

These imply that $B = 0$, and $A = \dfrac{j_\ell(kc)}{\gamma h_\ell^{1}(i\gamma c)}$; or

$$U_\ell(kr) = j_\ell(kr) + \frac{h_\ell^{(1)}(i\gamma r)}{h_\ell^{(1)}(i\gamma c)}\, j_\ell(kc). \tag{2.37}$$

In this approxiamtion, the contribution of the outer potential to the reaction matrix is,

$$\langle f_\ell | V | U_\ell \rangle = \int_c^\infty \{ j_\ell(kr) + j_\ell(kc) \left[\frac{h_\ell^{(1)}(i\gamma r)}{h_\ell^{(1)}(i\gamma c)} \right]^2 \} V(r) r^2 dr. \tag{2.38}$$

This is identical with the modified Born approximation of reference (20). This suggests that we take a trial wave function to be,

$$U_{\ell T} = j_\ell(kr) - \frac{h_\ell^{(1)}(ix_\ell r)}{h_\ell^{(1)}(ix_\ell c)} j_\ell(kc) \qquad (2.39)$$

where x_ℓ is a variational parameter that will depend on angular momentum and spin. U_ℓ has the form of a plane wave plus a term corresponding to the distortion from the hard core.

We have numerically integrated the quadratic functional $J(U_{\ell T})$ using a Gauss Quadrature integration formula. The contribution of the outer potential to the diagonal elements of the reaction matrix was determined by finding the minimum of J as a function of the parameter x_ℓ, using the Hamada-Johnstone nuclear potential (see page 45). The first iteration was started with the phase approximation for the reaction matrix in the energy denominator. The core contribution was calculated from equation (1.63). We then replaced the phase shift approximation for the reaction matrix in the denominator by the calculated G and repeated the procedure. This process converges quite rapidly for the density range $.1<k_F<2.7f$ as can be seen from the graph on page 42. The integrations have been done for the states $^3S_0$, $^3P_0$, $^3P_1$, $^3P_2$. The total energy per particle was found from equation (2.30). The numerical

results are summarized in table I on page 39 , with the graph of energy versus density on page 40 . We note imediately that nuclear forces alone are not sufficient to produce binding of the neutron gas for the range of densities considered. M. Barbi[23] has reached a similar conclusion for the three neutron problem by calculating the potential depths needed to bind three neutrons in various states using a numerical solution of the Euler-Lagrange variational equations. The depths obtained are not compatible with experiment so he concludes that the trineutron doesn't exist.

Since we make extensive use of the energy-density curve later on, it was found conveniant to express the results in polynomial form.

$$E/N = a_1 k_F^2 - \frac{a_2 k_F^2 + k_F^4}{a_3 k_F^2 + a_4} \qquad (2.40)$$

Here a_1 is just the kinetic energy term,

$$a_1 = \frac{3}{10}\frac{\hbar^2}{2m} = 12.449 \qquad a_2 = .17763$$

$$a_3 = .10199 \qquad a_4 = .15972$$

E will be in Mev if k_F is in F^{-1}.

Several other authors have calculated an equation of state for a neutron gas.[5,6,7,8] Sood and Moszkowski have treated the low density region ($k_F < .5F^{-1}$) by correcting the free pair G matrix for effects of the Pauli principle, assuming a separable potential. They also have found no evidence of binding by nuclear forces. Salpeter[21] treats the moderate density range ($.6F^{-1} < k_F < 1.9F^{-1}$) by extrapolating the Weizsacker semiempirical mass formula for nuclear matter to the case of a pure neutron gas. His values are lower than ours. Brueckner et. al.[7] have used the methods developed by Brueckner and Gammel[11] to evaluate self consistant single particle energies and total energy using a Gammel-Thaler potential. Their values are close to ours for the low to moderate range but ours fall below theirs for higher densities. Levinger and Simmons[8] have used perturbation theory to evaluate the energy of a neutron gas to first order for different velocity dependant potentials. Sprung et. al.[6] have used the Reid potential and Brueckner-Bethe many body theory to find the energy of a neutron gas and have also estimated the fractional density of protons in the system.

TABLE I

Results of Numerical Integration

	Units	Eq.	k_F=.1209F^{-1} .2602	.5615	.8856	1.2096	1.9678	2.606
$J(^1S_0)$:	F	2.5	-1.1208	-.8000	-.6075	-.5075	-.4613	-.3845	-.3563
$J(^3P_0)$:	F	2.5	...	-.0132	-.0425	-.0665	-.0795	-.0420	-.0015
$J(^3P_1)$:	F	2.5	...	+.00732	+.02377	+.03717	+.04521	+.0522	+.0600
$J(^3P_2)$:	F	2.5	...	-.00098	-.00453	-.0110	-.01847	-.04475	-.05225
G_{out} :	F	2.9 1.63	-14.083	-10.0041	-7.5568	-6.5028	-6.2516	-5.7162	-5.3749
G_{core} :	F	1.63	...	+.0017	+.0109	+.0363	+.0922	+.3956	+.8978
E/N	:Mev	2.30	+.1647	+.7214	+2.989	+6.616	+10.576	+19.739	+29.019

Figure 1.

Total Energy per Particle for the Neutron Gas

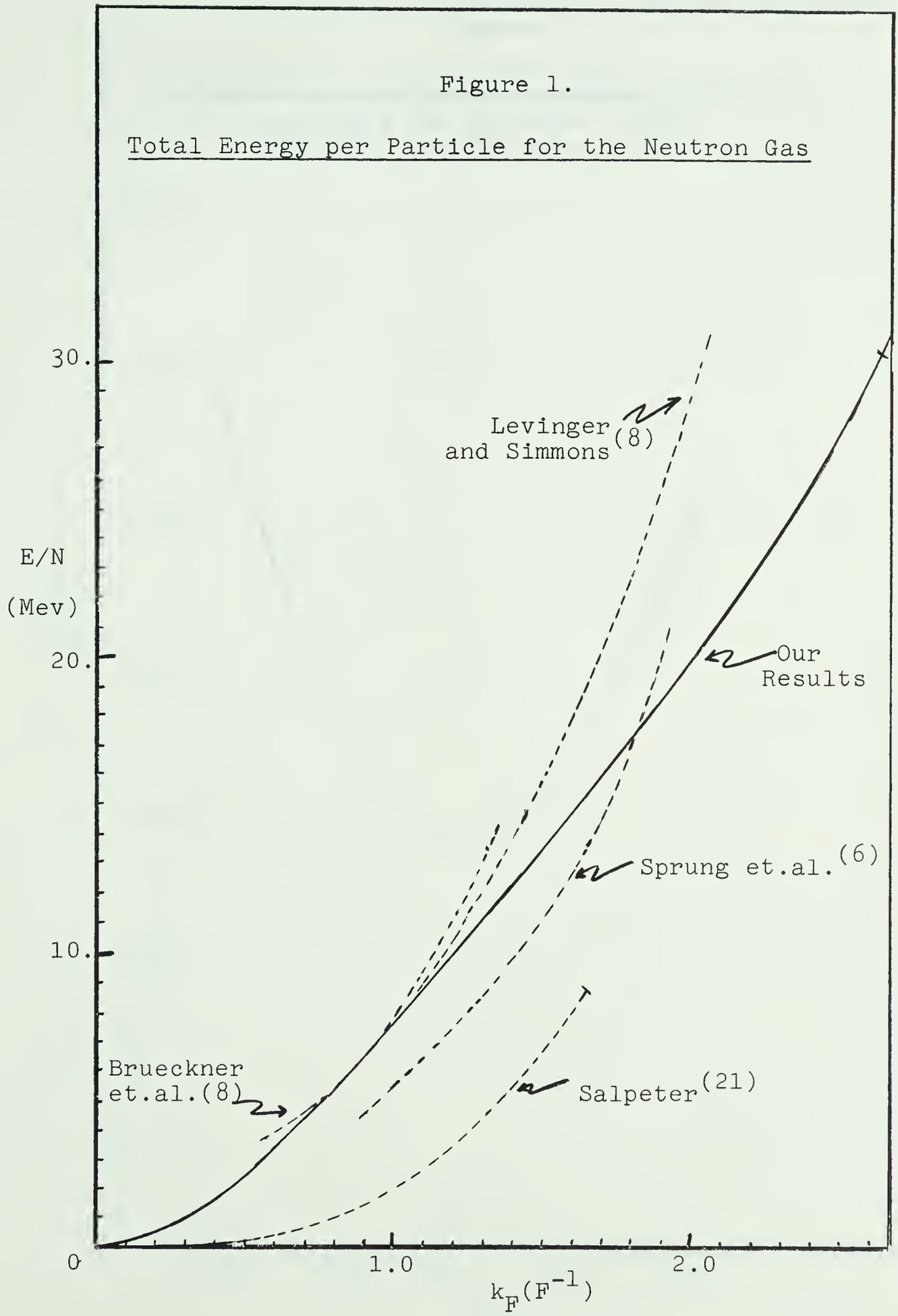
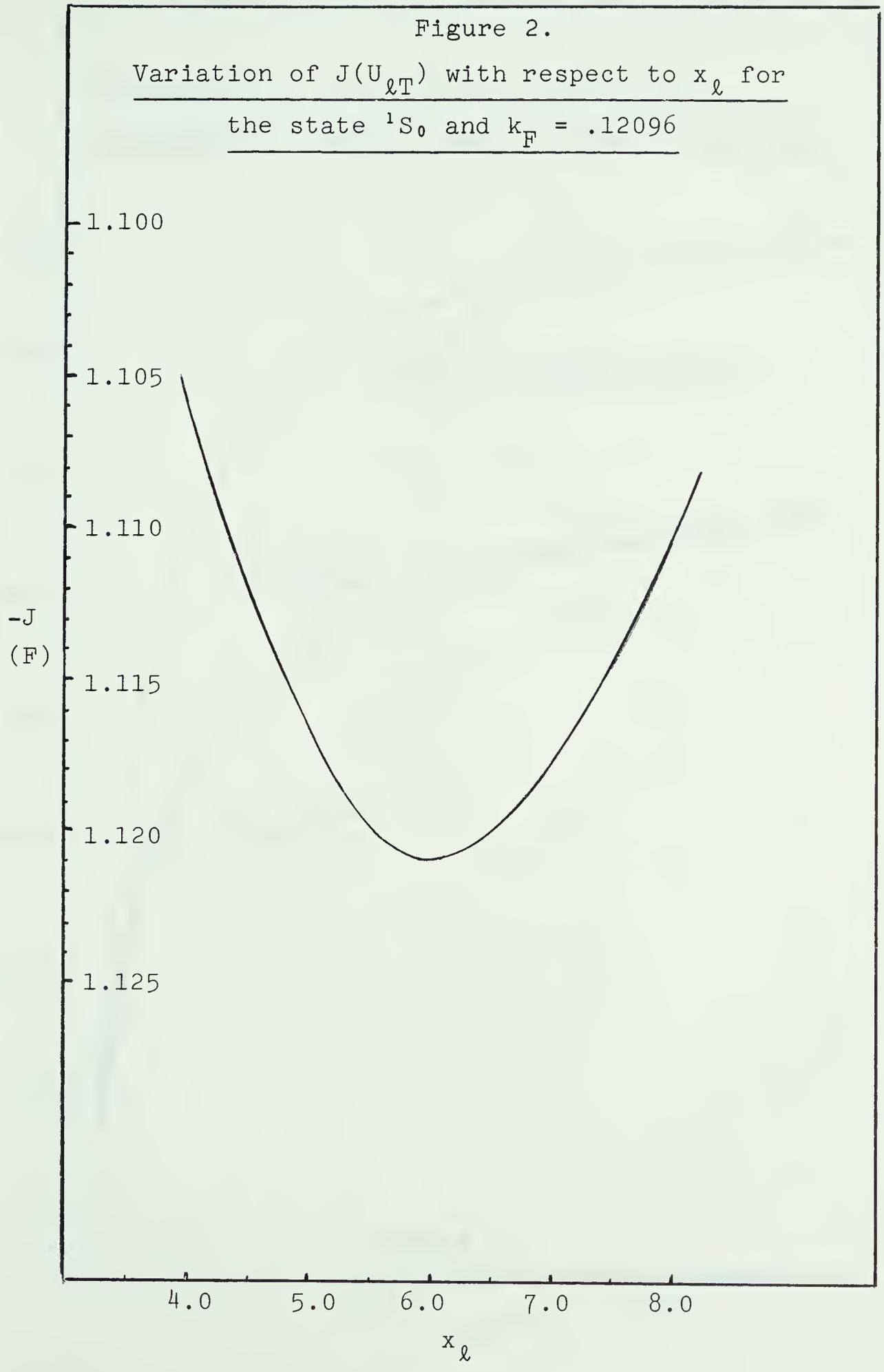


Figure 2.

Variation of $J(U_{\ell T})$ with respect to x_ℓ for the state $^1S_0$ and $k_F = .12096$



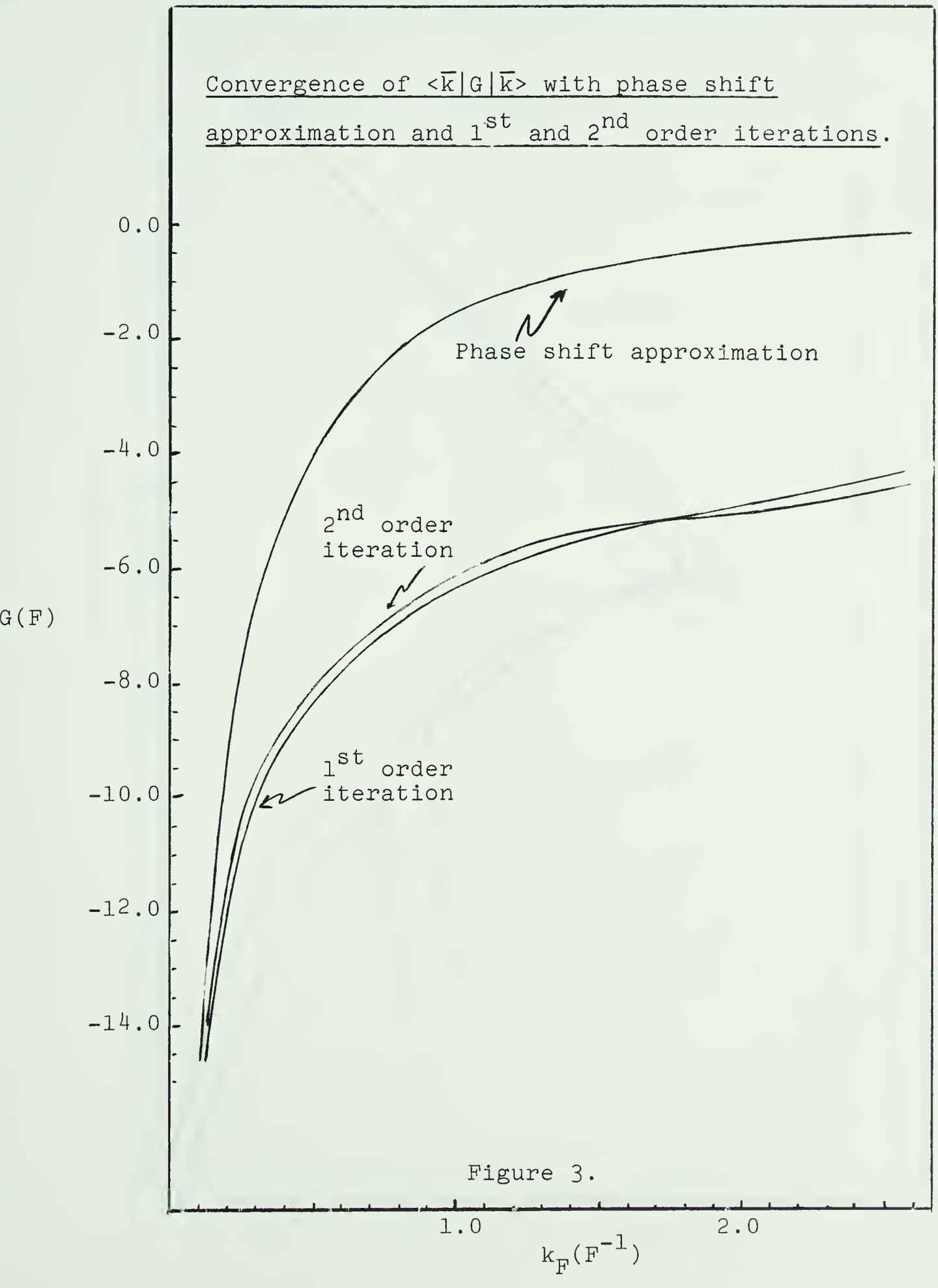

Convergence of <k̄|G|k̄> with phase shift approximation and 1st and 2nd order iterations.
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
-14.0
G(F)
Phase shift approximation
2nd order iteration
1st order iteration
1.0
2.0
kF(F-1)

Figure 3.

Figure 4. $\chi(r) = \frac{\sin(kr)}{k} - rU_{\ell}(r)$ for the $^1S_0$ wave.

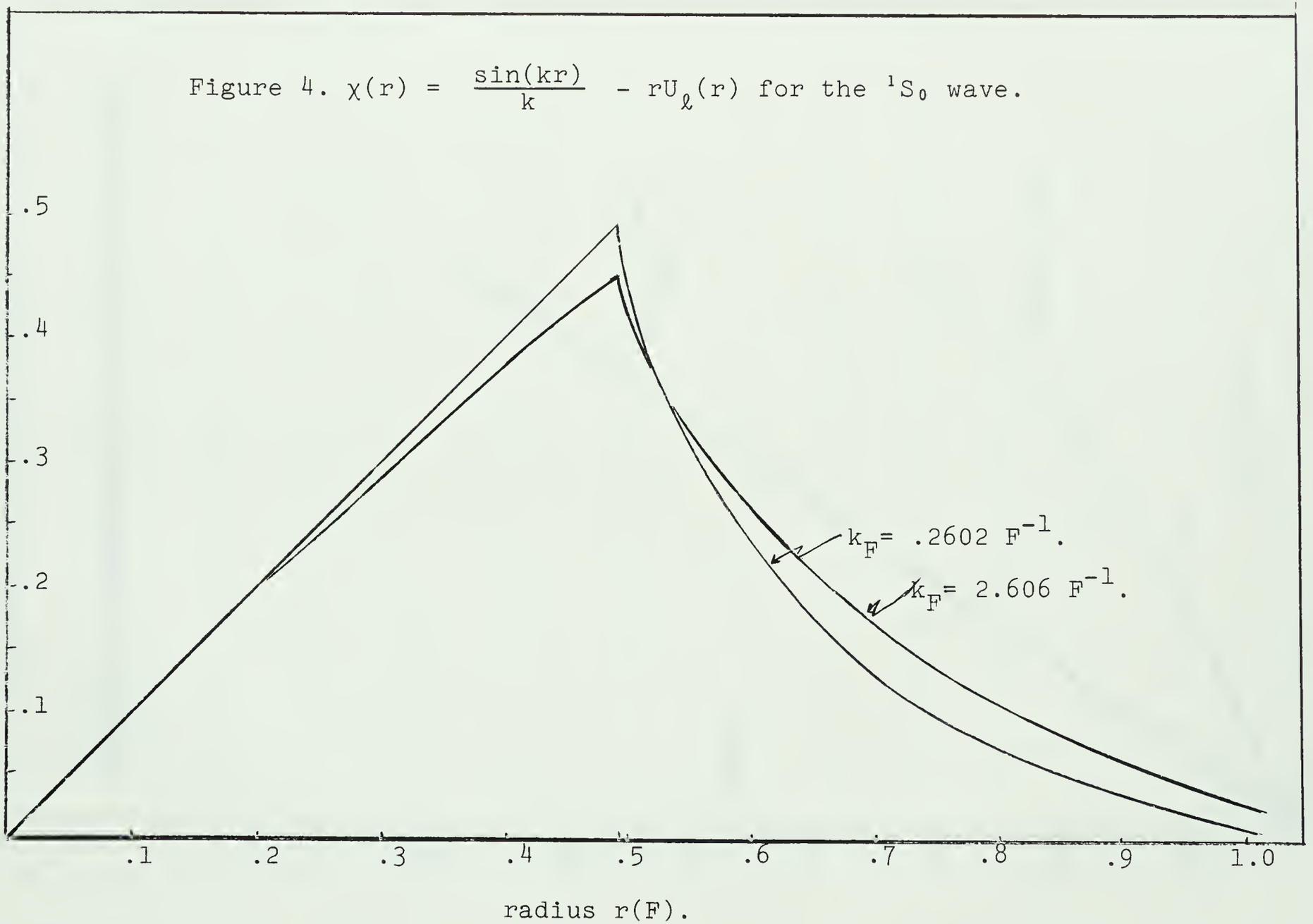
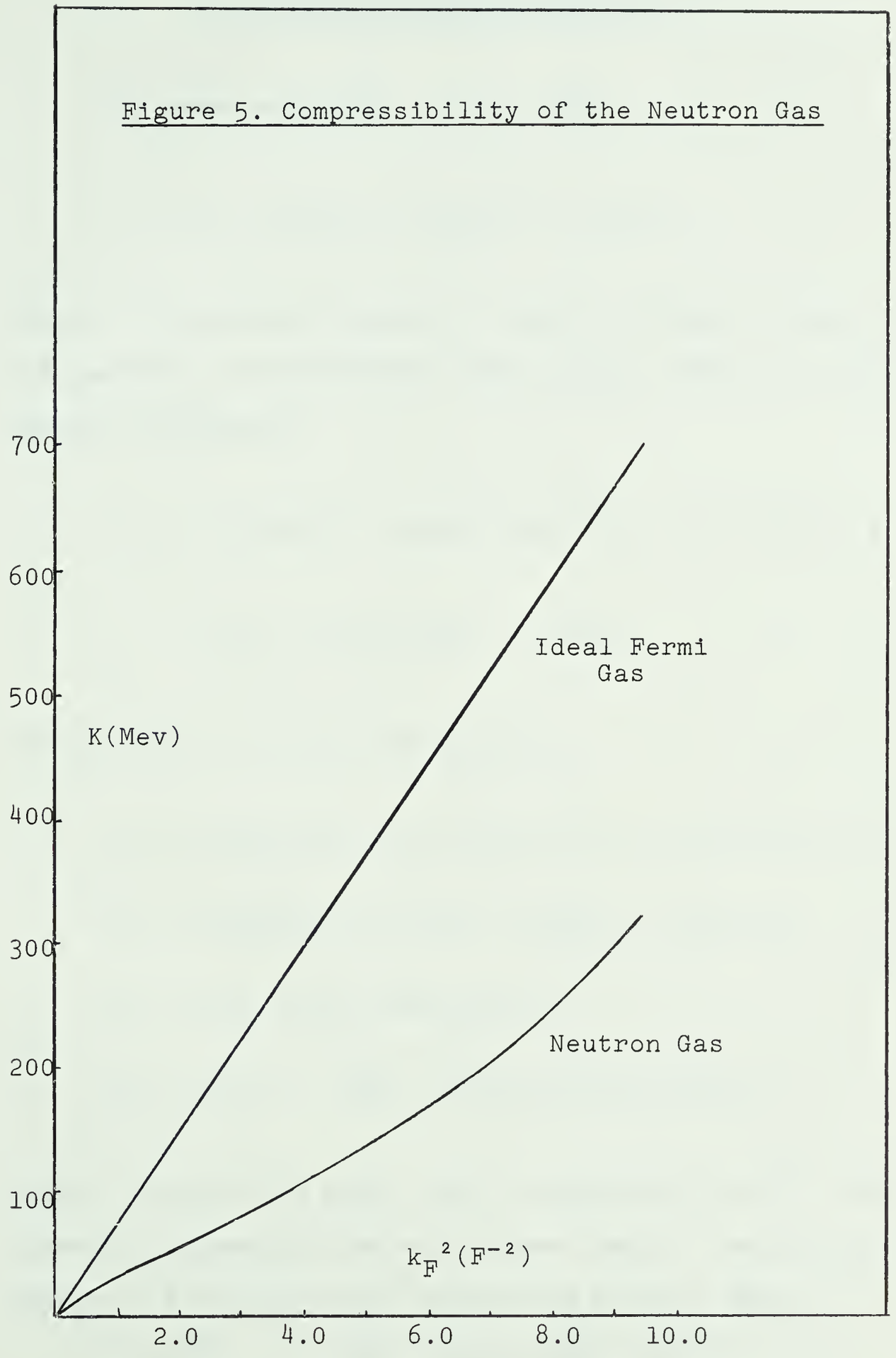


Figure 5. Compressibility of the Neutron Gas



The Hamada-Johnston Potential[1]

The model consists of four terms:

$$V = V_c + V_T S_{12} + V_{LS}(\vec{L}\cdot\vec{s}) + V_{LL} L_{12},$$

where C, T, LS and LL refer to central, tensor, linear LS and quadratic LS potentials respectively, and L_{12} is the operator defined by

$$L_{12} = (\vec{\sigma}_1\cdot\vec{\sigma}_2)\vec{L}^2 - \tfrac{1}{2}\{(\vec{\sigma}_1\cdot\vec{L})(\vec{\sigma}_2\cdot\vec{L}) + (\vec{\sigma}_1\cdot\vec{L})(\vec{\sigma}_2\cdot\vec{L})\}$$

$$= \{\delta_{LJ} + (\vec{\sigma}_1\cdot\vec{\sigma}_2)\}\vec{L}^2 - (\vec{L}\cdot\vec{S})^2.$$

The V_i(i=C, T, LS, LL) are given by,

$$V_c = 0.08(\tfrac{1}{3}\mu)(\vec{\tau}_1\cdot\vec{\tau}_2)(\vec{\sigma}_1\cdot\vec{\sigma}_2)Y(x)\ \{1+a_c Y(x)+b_c Y^2(x)\}$$

$$V_T = 0.08(\tfrac{1}{3}\mu)(\vec{\tau}_1\cdot\vec{\tau}_3)Z(x)\ \{1+a_T Y(x)+b_T Y^2(x)\}$$

$$V_{LS} = \mu G_{LS}\ Y^2(x)\ \{1+b_{LS} Y(x)\}$$

$$V_{LL} = \mu G_{LL}\ x^{-2}\ Z(x)\ \{1+a_{LL} Y(x)+b_{LL} Y^2(x)\}$$

where μ is the pion mass, and x is measured in μ^{-1}. The numerical parameters are on the next page. The hard core radius is x =0.343 in all states and μ=139.4 (Mev).

$$Y(x)= \frac{e^{-x}}{x} \qquad \text{and} \qquad Z(x)= \left(1+\frac{3}{x}+\frac{3}{x}\right)Y(x)$$

Parameters of the Hamada-Johnston Potential

state	a_c	b_c	a_T	b_T	G_{LS}	b_{LS}	G_{LL}	a_{LL}	b_{LL}
singlet even	8.7	10.6	...	...	...	...	-.000891	.2	-.2
triplet odd	-9.07	3.48	-1.29	.55	.1961	-7.12	-.000891	-7.26	6.92
triplet even	6.0	-1.0	-0.5	.2	.0743	-.1	.00267	1.8	-.4
singlet odd	-8.0	12.0	...	...	...	...	-.00267	2.0	6.0

Chapter IV

The Relativistic Equations of Hydrostatic Equilibrium

To date, numerous theoretical models of the origin and evolution of stars and galaxies have been proposed. These may be considered generally as falling into two groups. In one, the initial stage consists of diffuse galactic matter (mostly hydrogen) which eventually tends to condense with the lost gravitational energy radiated in the form of light and heat. The temperature of the interior of the mass rises until nuclear fusion of hydrogen to helium occurs. The next period in the life of the star is the so called normal period during which all the stellar thermonuclear processes occur and is also the longest stage in the life of the star. Eventually there must be exhaustion of the nuclear fuel and contraction of the star to denser states. What happens next depends specifically on the configuration considered and will be discussed later.

In the second group of evolutionary models the initial state is a superdense body with the normal course of evolution proceding to states of lower density. It is not our purpose here to discuss the validity of these models but only to indicate that in either case the question of matter at higher than normal density arises.

It has been suggested that a system of baryons at the endpoint of thermonuclear evolution form Fe^{56} in a body centered lattice.(24) As the number of particles

increases, the gravitational forces increase and relativistic electrons appear. When densities exceed about 3×10^{11} gm/cc all of the heavy ions become unstable against disintegration to neutrons by means of electron capture[24] and the system in equilibrium consists mainly of neutrons contaminated with some protons and electrons. Near and above 10^{15} gm/cc mesons and other baryons appear.[10] For the range 10^{12} gm/cc to 10^{15} gm/cc we are dealing with a degenerate gas which may be considered practically as consisting only of neutrons. Cameron[10] has considered the fraction of protons and electrons for these densities and has shown them to be about three orders of magnitude lower than the number of neutrons.

Many works have predicted the existence of neutron stars. Recent discoveries of galactic sources of X-rays renewed hope in the theories, however occultation experiments have shown most of these are too large to be neutron stars. As early as 1939, Oppenheimer and co-workers[25] studied stars during the last stages of active life when sources of internal energy were exhausted and the star had cooled down. They predicted equilibrium configurations of neutron stars at zero temperature by assuming that the particles form an ideal gas. For masses greater than about .7 solar mass there were no equilibrium solutions. When thermonuclear sources are exhausted heavier stars will contract unobstructed unless they can loose mass by some mechanism. It is now generally accepted that supernovae explosions are triggered

by collapse of stellar cores followed by sudden release of gravitational energy or ignition of nuclear fuel in the remaining envelope.

More recent authors have studied these problems taking into account in various ways the interparticle interactions and allowing for the formation of hyperons at higher densities.[10,26,27,28] They have confirmed the result that there exists an upper limit around one solar mass for the mass of a neutron star beyond which there is no stable configuration.

The behavior of nuclear forces at extremely high energies is known only approximately and the interaction of hyperons is not known at all. Consequently it is difficult to discuss the effect that formation of hyperons has on superdense configurations. At densities around 10^{17}gm/cc the interparticle distance is about 4×10^{-12}cm. or about the radius of the hard core for nucleons. Since the interparticle distance is of the order of 10^{-13}cm or less, any equations should include the structure of elementary particles themselves. Bahcall and Wolf[29] suggest that the concept of distinct strongly interacting particles is not meaningful for densities greater than about eight times nuclear densities. Also the distinction between Fermions and Bosons probably disappears around this density. In this case the star should be considered as a whole and not as consisting of individual particles. It seems likely now that matter should follow a simple asymtotic equation of

state at higher densities.[31,30] We will assume this type of behavior at high densities and attempt to describe the low to moderate density range correctly.

In the case of out own sun, the ratio of the gravitational radius to the actual radius is small so relativistic effects are small. In the case of superdense stellar configurations however, these two quantities may be comparable. In other words the high concentration of mass produces a large curvature of space-time near the star. The basis of our calculations must be Einstein's gravitational equations which relate the energy-momentum tensor $T_{\mu\nu}$ to the fundamental metric tensor $g_{\mu\nu}$. In the system of units for which the velocity of light and the constant of gravitation are equal to unity, the field equations are;

$$-8\pi T_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}R\, g_{\mu\nu} + \Lambda g_{\mu\nu} \qquad (4.1)$$

where $R_{\mu\nu}$ is the contracted Riemann-Christoffel tensor and R is its trace. The effects of the last term are insignificant within the size of the solar system so we may consider Λ (the cosmological constant) equal to zero.

Physically we expect a static spherical distribution of matter so we choose spatial co-ordinates r, θ, ϕ and time co-ordinate t to reflect this symetry. The most general static line element exhibiting spherical symetry may be expressed in the form[32],

$$ds^2 = -e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 + e^{\nu}dt^2 \qquad (4.2)$$

where λ and ν are functions of r alone. In other words the components of the metric tensor are;

$$\begin{aligned} g_{11} &= -e^{\lambda} & g_{22} &= -r^2 \\ g_{33} &= -r^2\sin^2\theta & g_{44} &= e^{\nu} \end{aligned} \qquad (4.3)$$

$$g_{\mu\nu} = g^{\mu\nu} = 0 \quad \text{for } \mu \neq \nu \text{ and } g^{jj} = \frac{1}{g_{jj}} \quad \text{(no sum)}.$$

The contracted Riemann-Christoffel tensor $R_{\mu\nu}$ can be written in the form;

$$R_{\mu\nu} = \frac{-\partial}{\partial x^{\alpha}}\{\mu\nu,\alpha\} + \{\mu\alpha,\beta\}\{\nu\beta,\alpha\} + \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\log\sqrt{-g} - \{\mu\nu,\alpha\}\frac{\partial}{\partial x^{\alpha}}\log\sqrt{-g}. \qquad (4.4)$$

where $g = |g_{\mu\nu}|$ and the Christoffel three-index symbols with curly brackets are defined by;

$$\{\mu\nu,\sigma\} = \tfrac{1}{2}g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right). \qquad (4.5)$$

For example;

$$\{11,1\} = \tfrac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}g_{11}\left(\frac{\partial g_{11}}{\partial r}\right) = \frac{-1}{2e^{\lambda}}\frac{\partial}{\partial r}(-e^{\lambda}) = \frac{\lambda'}{2}$$

The remaining ones are got in a similar manner, making use of the relation $\{\mu\nu,\sigma\} = \{\nu\mu,\sigma\}$. The non-vanishing symbols corresponding to our form of line element are; (Equations 95.5 of reference 32)

$$\begin{aligned}
\{11,1\} &= \tfrac{1}{2}\lambda' & \{31,3\} &= 1/r \\
\{12,2\} &= 1/r & \{32,3\} &= \mathrm{ctn}\theta \\
\{13,3\} &= 1/r & \{33,1\} &= -r\sin^2\theta e^{-\lambda} \\
\{14,4\} &= \tfrac{1}{2}\nu' & \{33,2\} &= -\sin\theta\cos\theta \\
\{21,2\} &= 1/r & \{41,4\} &= \tfrac{1}{2}\nu' \\
\{22,1\} &= -re^{-\lambda} & \{44,1\} &= \tfrac{1}{2}e^{\nu-\lambda}\nu' \\
\{23,3\} &= \mathrm{ctn}\theta. &&
\end{aligned} \qquad (4.6)$$

Using these and equation (4.4), the components of $R_{\mu\nu}$ may be calculated; eg.

$$R_{11} = \frac{-\partial}{\partial r}\{11,1\} + \{11,1\}^2 + \{12,2\}^2 + \{13,3\}^2 + \{14,4\}^2$$

$$+ \frac{\partial^2}{\partial r^2}\log\sqrt{-g} - \{11,1\}\frac{\partial}{\partial r}\log\sqrt{-g}\ .$$

$$\log\sqrt{-g} = \tfrac{1}{2}\log(e^{\lambda+\nu} r^4\sin^2\theta)$$

$$\frac{\partial}{\partial r}\log\sqrt{-g} = \tfrac{1}{2}(\lambda'+\nu') + \frac{2}{r}$$

$$\frac{\partial^2}{\partial r^2} \log\sqrt{-g} = \tfrac{1}{2}(\lambda'' + \nu'') - \frac{2}{r^2}$$

$$R_{11} = \frac{\nu'^2}{4} + \frac{\nu''}{2} - \frac{\lambda'\nu'}{4} - \frac{\lambda'}{r} . \tag{4.7}$$

Similarly,

$$R_{22} = e^{-\lambda} + re^{-\lambda}\left(\frac{\nu'-\lambda'}{2}\right) - 1 \tag{4.8}$$

$$R_{33} = r\sin^2\theta\, e^{-\lambda}\, \tfrac{1}{2}(\nu'-\lambda') - \sin^2\theta\,(1-e^{-\lambda}) \tag{4.9}$$

$$R_{44} = \tfrac{1}{2}e^{\nu-\lambda}\left(-\nu'' + \frac{\nu'\lambda'}{2} - \frac{\nu'^2}{2} - \frac{2\nu'}{r}\right). \tag{4.10}$$

It will prove simpler to express the field equations in mixed tensor form;

$$-8\pi T^{\mu}_{\nu} = R^{\mu}_{\nu} - \tfrac{1}{2}R\, g^{\mu}_{\nu} . \tag{4.11}$$

Now $g^{\mu}_{\nu} = \delta^{\mu}_{\nu}$ and $T^{\mu}_{\nu} = g^{\mu\alpha}T_{\alpha\nu}$. Remembering that $g^{\mu\nu} = 0$ if $\mu \neq \nu$, (4.11) becomes,

$$-8\pi T^{\mu}_{\nu} = g^{\mu\mu}\, R_{\mu\nu} - \tfrac{1}{2}R\, \delta_{\mu\nu}. \tag{4.12}$$

The non-vanishing components of T^{μ}_{ν} are T^1_1, T^2_2, T^3_3, T^4_4, which may be found by substituting the previous relations as follows.

$$-8\pi T_1^1 = g^{11}R_{11} - \tfrac{1}{2}(g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44})$$

or, $$-16\pi T_1^1 = \frac{R_{11}}{g_{11}} - \frac{R_{22}}{g_{22}} - \frac{R_{33}}{g_{33}} - \frac{R_{44}}{g_{44}} .$$

$$-16\pi T_1^1 = -e^{-\lambda}\left(\frac{\nu'^2}{4} + \frac{\nu''}{2} - \frac{\lambda'\nu'}{4} - \frac{\lambda'}{r} \right)$$

$$+ \frac{1}{r^2}\left[e^{-\lambda} + re^{-\lambda}\tfrac{1}{2}(\nu'-\lambda') - 1\right] + \frac{1}{r^2\sin^2\theta}\left[r\sin^2\theta\, e^{-\lambda}\tfrac{1}{2}(\nu'-\lambda')\right.$$

$$\left. -\sin^2\theta(1-e^{-\lambda})\right] - \tfrac{1}{2}e^{-\nu}\left[e^{\nu-\lambda}\tfrac{1}{2}(-2\nu'' + \nu'\lambda' - \nu'^2 - \frac{4\nu'}{r})\right].$$

$$8\pi T_1^1 = -e^{-\lambda}\{\frac{\nu'}{r} + \frac{1}{r^2}\} + \frac{1}{r^2} \qquad (4.12)$$

Similarly, $$8\pi T_2^2 = -e^{-\lambda}\{\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r}\} \qquad (4.13)$$

$$T_3^3 = T_2^2 \qquad (4.14)$$

$$8\pi T_4^4 = e^{-\lambda}\left(\frac{\lambda'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \qquad (4.15)$$

Assuming the properties of a perfect fluid (ie. a mechanical medium exerting no transverse stresses), Tolman has shown (Reference 32, equation 85.7) that we can express the energy-momentum tensor in the general form;

$$T^{\mu\nu} = (\rho_0 + p_0)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} - g^{\mu\nu}p_0$$

or, $$T_\mu^\nu = (\rho_0 + p_0)\, g_{\alpha\mu}\frac{dx^\alpha}{ds}\frac{dx^\nu}{ds} - g_\mu^\nu p_0 \qquad (4.16)$$

where ρ_0 and p_0 are the macroscopic energy density and hydrostatic pressure respectively measured in proper co-ordinates. For the static case considered here, the components of fluid velocity are;

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0, \qquad \frac{dt}{ds} = e^{-\nu/2}.$$

Putting these into (4.16) we obtain;

$$T^1_1 = T^2_2 = T^3_3 = -p_0 \ , \qquad T^4_4 = (\rho_0+p_0)\, e^{\nu}e^{-\nu} - p_0 = \rho_0.$$

We now have; (droping the subscripts on p_0 and ρ_0)

$$8\pi p = e^{-\lambda}\{\frac{\nu'}{2} + \frac{1}{r^2}\} - \frac{1}{r^2}$$

$$8\pi p = e^{-\lambda}\{\frac{\nu'}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r}\} \qquad (4.17)$$

$$8\pi \rho = e^{-\lambda}\{\frac{\lambda'}{r} - \frac{1}{r^2}\}+ \frac{1}{r^2}$$

Equate the first two equations in (4.17) to get;

$$e^{-\lambda}\{\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu'-\lambda'}{2r} - \frac{\nu'}{r} - \frac{1}{r^2}\}+ \frac{1}{r^2} = 0. \qquad (4.18)$$

Multiplying be 2/r and rearranging, this is equivalent to;

$$\frac{2}{r^3}+e^{-\lambda}\{\frac{\nu''}{r} - \frac{\nu'}{r^2} - \frac{2}{r^3}\} -e^{-\lambda}\{\frac{\nu'}{r} + \frac{1}{r^2}\}\lambda' + e^{-\lambda}\{\frac{\lambda'}{r} + \frac{\nu'}{r}\} \frac{\nu'}{2} = 0. \quad (4.19)$$

Comparing this to (4.17) allows us to write (4.19) as;

$$\frac{dp}{dr} + (\rho+p)\frac{\nu'}{2} = 0. \quad (4.20)$$

Einstein's field equations reduce to the following three equations;

$$8\pi p = e^{-\lambda} \{\frac{\nu'}{r} + \frac{1}{r^2}\} - \frac{1}{r^2}$$

$$8\pi\rho = e^{-\lambda} \{\frac{\lambda'}{r} - \frac{1}{r^2}\}+ \frac{1}{r^2} \quad (4.21)$$

$$\frac{dp}{dr} = -\frac{(p+\rho)\nu'}{2}.$$

If we can determine λ and ν as functions of r, the equilibrium of the system will be determined by adding some equation of state $\rho = \rho(p)$. In the empty space surrounding the spherical distribution of matter $p=\rho=0$ and Schwarzschild's exterior solution of Einstein's field equations is valid.(33)

$$e^{-\lambda} = 1 + A/r, \qquad e^{\nu} = B(1 + A/r)$$

A and B are fixed by the requirement that the components

of the metric tensor go over into their weak field form at large distances. ie. B = 1, A = -2m where m is the total Newtonian mass of the matter as calculated by a distant observer.

The boundary of the sperical distribution is at $r = r_b$ where $p = 0$ (remember that $p>0$ for $r<r_b$). Using the equation of state $\varepsilon = \varepsilon(p)$, the last equation in (4.21) may be integrated.

$$\nu(r) = \nu(r_b) - \int_0^{p(r)} \frac{2dp}{p + \rho(p)}, \quad \text{or} \quad e^{\nu(r)} = e^{\nu(r_b)} \exp\left[- \int_0^{p(r)} \frac{2dp}{p+\rho(p)} \right] \tag{4.23}$$

Now e^{ν} must be continuous across the boundary. From (4.22)

$$e^{\nu(r)} = \left(1 - \frac{2m}{r_b}\right) \exp\left[- \int_0^{p(r)} \frac{2dp}{(p+\rho)} \right] \tag{4.24}$$

Introducing a new variable $u(r) = \frac{1}{2}r(1-e^{-\lambda})$ or $e^{-\lambda} = 1 - \frac{2u}{r}$, the second equation in (4.21) may be written,

$$\frac{du}{dr} = 4\pi\rho(r)\, r^2 \tag{4.25}$$

The first equation in (4.21) may be re-expressed by putting $e^{-\lambda} = \left(1 - \frac{2u}{r}\right)$ and using $\nu' = \frac{dp}{dr}\left(-2/(p+\rho)\right)$ from the third equation.

$$\frac{dp}{dr} = \frac{-(p+\rho)\{4\pi pr^3 + u\}}{r(r-2u)} \qquad (4.26)$$

The value of $e^{\lambda(r)}$ must be continuous across the boundary.

$$u(r_b) = \frac{r_b}{2}\,[1-e^{-\lambda(r_b)}] = \frac{r_b}{2}[1-(1-\frac{2M}{r_b})] = M$$

$u(r_b)$ is the observable mass of the star as determined by its gravitational effect on a distant test particle. At this point we make a change in notation in order to avoid confusion later on. In the limit of large density we have to distinguish between the total energy (mass) density, (which includes the sum of particle masses plus energy of motion and interaction, not including gravitational energy) and the density of the rest mass of the particles. The total macroscopic energy density previously labled ρ in this chapter will from this point be called ε. The density of the rest mass of the particles will be $\rho' = m\rho(k_F)$ where $\rho(k_F)$ is the number density of particles defined in equation (1.9), and m is the rest mass of the neutron. The system will then be described by;

$$\frac{du(r)}{dr} = 4\pi\varepsilon r^2$$

$$\frac{dp}{dr} = \frac{-(p+\varepsilon)[\,4\pi pr^3 + u]}{r(r-2u)} \qquad (4.27)$$

$$\varepsilon = \varepsilon(p)$$

The initial conditions will have the form;

$$u(0) = 0, \quad \varepsilon = \varepsilon(0) \equiv \varepsilon_c, \quad p = p(\varepsilon(0)) \tag{4.28}$$

where ε_c is the total energy density at the center of the configuration. Equations (4.27) are integrated simutaneously to the point $r = r_b$ where $p = 0$.

The proper distance in a gravitational field is determined from

$$dr_p = \sqrt{-g_{rr}}\, dr = \left(1 - \frac{2u}{r}\right)^{-1/2}. \tag{4.29}$$

The proper mass is therefore

$$\frac{dM_p}{dr} = 4\pi\rho' r^2 \left(\frac{r}{r-2u}\right)^{1/2}. \tag{4.30}$$

The proper mass is the mass the star would have if the particles were dispersed to infinity. The binding energy in mass units is the proper mass minus the gravitational mass and is given by;

$$\frac{dM_b}{dr} = 4\pi r^2 \left(\rho' \left[\frac{r}{r-2u}\right]^{1/2} - \varepsilon\right). \tag{4.31}$$

Chapter V. The Maximum Stable Mass of a Neutron Star

We have already obtained the total energy per particle of the neutron gas (E) as a function of density (or k_F) in equation (2.40). The pressure of the gas is given by

$$p = \rho^2\left(\frac{\partial E}{\partial \rho}\right) = 1.80351 \times 10^{31}\left(\frac{\partial E}{\partial k_F}\right) k_F^4 \text{ dynes/cm}^2. \quad (5.1)$$

where E is in Mev and k_F in F^{-1}. The total mass density ε is defined by

$$\varepsilon = [\rho E(\rho) + \rho' c^2] = [\, E(\text{Mev}) + 939.5]\, k_F^3 \times 0.006011 \times 10^{13} \text{g/cc} \quad (5.2)$$

where k_F is again measured in F^{-1}. Remembering that we are considering a static structure with spherical symetry and also that p and ε are in proper co-ordinates, the stress energy tensor $T_{\mu\nu}$ may be written,

$$T_{\mu\nu} = \begin{pmatrix} -p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix} \quad (5.3)$$

Landau and Liftshitz[30] state that for a non-interacting

gas the trace of $T_{\mu\nu}$ must be positive, ie.

$$p \leqslant \frac{\varepsilon}{3} . \qquad (5.4)$$

The equality sign would be valid only for a photon gas. However, Zel'dovich[31] has constructed a relativistically invariant example of interaction of particles with a field of heavy neutral vector mesons which yields $p \to \varepsilon$ at higher densities. In this case the velocity of sound, which is given by

$$v_s = c\sqrt{\frac{dp}{d\varepsilon}} \qquad (5.5)$$

is always less than or equal to c, the velocity of light. This seems more aesthetically satisfying than the previous example which yields

$$v_s \leqq \frac{c}{3} . \qquad (5.6)$$

To make our equation of state have the correct asymtotic behavior at high densities ($p \to \varepsilon$), it was found convenient to modify equation (2.40) as

$$E = \left[a_1 k_F^2 - \left[\frac{a_2 k_F^2 + k_F^4}{a_3 k_F^2 + a_4} \right] \right] \chi(k_F) + [\, 1 - \chi(k_F)\,] k_F^3 \qquad (5.7)$$

where $$\chi(k_F) = \left(1 + \exp[\, k_F - 8 \,] \right)^{-1} \qquad (5.8)$$

This leaves the low to moderate density regions unchanged but at $\varepsilon \simeq 10^{16}$g/cc, $p \to \varepsilon$. Ours then is a parametric equation of state,

$$\begin{aligned} \varepsilon &= \varepsilon(k_F) \\ p &= p(k_F) \end{aligned} \qquad (5.9)$$

which may be compared in Figure 6 with those of Cameron[34]. We have used this equation of state to integrate the equations of equilibrium (4.27), (4.30), and (4.31) for various central energy densities. The results are summarized in table III and Figure 7.

The quantities in equations (4.27), (4.30), and (4.31) are in relativistic units ($c = G = 1$ and $M^4c^5/32\pi^2\hbar^3 \equiv K = 1/4\pi$). In order to change back to c.g.s. units we must multiply by the following conversion factors.[34]

$$\begin{aligned} &\text{length:} \quad r_0 = 2\sqrt{2\pi}(\hbar/Mc)^{\frac{3}{2}}\,(c/\sqrt{GM}) \\ &\qquad\qquad = 13.7 \text{ km} \simeq 2 \times 10^{-5} R_\odot \qquad (5.10) \\ &\text{mass:} \quad m_0 = r_0c^2/G = 1.85 \times 10^{34}\text{g} = 9.29\, M_\odot \\ &\text{pressure:} \quad p_0 = (M^4c^5/32\pi^2\hbar^3)4\pi = 6.46 \times 10^{36}\text{dynes/cm}^2 \\ &\text{density:} \quad \rho_0 = p_0/c^2 = 7.15 \times 10^{15}\text{g/cm}^3. \end{aligned}$$

The neutron gas is relativistic in the core of some of the configurations considered. Strictly speaking we should have used the relativistic expression for the kinetic energy,

$$K.E. = c(m^2c^2+p^2)^{1/2} \qquad (5.11)$$

instead of

$$K.E. = (mc^2+ p^2/2m) \qquad (5.12)$$

Saakyan[35] has integrated the equations of equilibrium using equations of state for real neutron gases with either (5.11) or (5.12) and has shown that the mass radius values agree within three significant figures. The reason for this is that the repulsive forces become dominant at energies where the neutron gas is relativistic so it makes little difference which of (5.11) of (5.12) is chosen.

The first qualitative feature that appears when we plot the observable mass against the central energy density (Figure 7) is the existence of a maximum in the curve at $\varepsilon_c = 6.3 \times 10$ g/cm and $M = 1.6M_\odot$. This mass maximum is commonly called the OV (Oppenheimer and Volkoff) maximum. As previously mentioned, this maximum was first noticed by Oppenheimer and Volkoff in 1939 when they integrated the equilibrium equations using the equation of state for an ideal Fermi gas.

The appearance of a maximum in the curve is not unexpected since it is easy to show that the mass of an equilibrium configuration cannot be arbitrarily large. It is well known that the gravitational force on a test mass goes to infinity if

$$r \to r_g = \frac{2GM}{c^2} \tag{5.13}$$

Here r_g is called the gravitational radius, G is the gravitational constant and M is the observable mass of the configuration considered. Define an average energy density

$$\varepsilon = \frac{M}{\frac{4}{3}\pi R^3} \tag{5.14}$$

then $R > r_g$ implies $M < \dfrac{c^3}{(2G)^{3/2}(\frac{4}{3})^{1/2}\varepsilon^{1/2}}$ (5.15)

The largest possible mass of the equilibrium configuration must decrease with increasing ε.

Zel'dovich[36] has done an analysis of the exact equilibrium conditions for the star and has shown that a configuration is stable if

$$\frac{dM}{d\varepsilon_c} > 0. \tag{5.16}$$

This is a natural criterion since in the stable state, the

the addition of mass causes compression with an increase in pressure to compensate the gravitational force.

Neutron stars whose observable mass is less than 1.6 $M_\odot$ will be dynamically stable. The star as a whole will not consist only of neutrons since in the outer regions the pressure is not sufficient for the existance of stable neutrons. It was apparent during the integrations that the density in the interior of the star remained remarkably constant for the greatest part of the star radius and then fell rapidly to zero. This is the same observation that Cameron[34] has made. He has shown that models of low density may have extensive electronic-nuclear envelopes but for higher central energy densities (10^{14}g/cm^3) the shells diminish and are no longer important. Cameron also has integrated the equilibrium equations for pure neutron configurations as well as for models in which the composition changes with density. The effect of changes in composition on the mass radius values is very small compared to the effect that different assumptions about nuclear forces has. We feel completely justified in regarding the star (for the range $10^{14} < \varepsilon_c < 10^{16}$g/cm) as being composed of a neutron core surrounded by thin or negligible envelopes of electrons and nuclei. Chiu and Salpeter[37] have determined the surface temperature and luminosity for neutron stars. The surface temperature will fall in the range $10^6 \rightarrow 10^7$ ^{0}K. For temperatures even as high as 10^9^{O}K and $\varepsilon_c = 10^{15}$g/cm^3

the ratio of Fermi energy to thermal energy is about 400 so the cold matter approximation is justified. For temperatures any higher than this neutrino processes would rapidly dissipate energy. Also the high thermal conductivity of the degenerate neutron gas and small dimensions of the neutron star insure that the temperature is low. The luminosity is thousands of times greater than our sun but the maximum of the energy distribution lies in the X-ray region. Most of the radiation will not penetrate our atmosphere so discovery of a neutron star may have to wait for an orbiting telescope. Chiu[9] has said that in order to be visible on the surface of the earth by the largest telescope today, a neutron star would have to be within our own solar system. It is no small wonder that a neutron star has not been discovered to this date.

At this point we must say something about the question regarding the behavior of configurations with masses greater than the OV maximum for which there are no stable solutions to the relativistic equations. To either side of the OV maximum the gravitational force exceeds the pressure and causes the star to contract. To the low density side however, the pressure increases and returns the star to equilibrium while to the high density side the forces take the star further from equilibrium and the star contracts more. When an amount of cold matter is assembled with a mass exceeding the critical figure of 1.6 $M_{\odot}$, then unless

it can shed mass by some mechanism it will become unstable against collapse and contract unobstructed. This is the phase known as gravitational collapse. It is beyond the scope of the present work to attempt any calculations on these processes but for the sake of completness we will briefly mention some of the current ideas. M. A. Podurets[38] has shown that the pressure forces are not important for calculating the dynamics of collapse. For a first approximation it is possible to consider the free fall of a particle in a Schwartzchild field. Zel'dovich and Novikov[39] explain the process as follows. A remote observer sees the star collapsing at a large rate when its radius is still much larger than r_g. The velocity of collapse goes to zero as $r \to r_g$ (as seen by the remote observer). The time taken to fall to r_g as seen by a local observer is finite however and the star continues to collapse after reaching r_g. At $r = r_g$ the Schwarzchild singularity occurs. On this surface the curvature of space becomes infinite (no light emitted from this surface will reach us). As seen by the external viewer, when the radius reaches r_g the star stops radiating almost instantaneously (although the star still interacts with surrounding bodies through its own gravitational field). The star has effectively cut itself off from the rest of the universe by producing such a large curvature that not even photons may leave it.

At the International Symposium on Gravitational Collapse and Relativistic Astrophysics in Dallas (1963) it was concluded that there was no principle which could prevent the collapse from proceeding to completion and in the process the constituent baryons would disappear. This is a violation of present day ideas and may represent the appearence of new physical principles or may in fact indicate that we must find some other approach to describe the collapse. At any rate the problem as it now stands is unresolved.

Table III. Characteristics of Neutron Star Models.

$\log \varepsilon_c (g/cm^3)$	$\log p_c (\frac{dynes}{cm^2})$	Radius(km.)	$\frac{M}{M_\odot}$	$\frac{M_p}{M_\odot}$	$\frac{M_b}{M_\odot}$	ϕ = (red shift)
12.8500	29.926	27.809	.04751	.04755	.00003	.00321
13.2893	30.645	23.152	.06778	.06787	.00008	.00430
13.755	31.732	18.766	.09043	.09061	.00018	.00708
14.287	32.624	14.316	.11087	.11727	.00039	.01200
14.664	33.532	11.105	.15342	..1544	.00102	.02030
14.958	34.166	8.8908	.24566	.25005	.00439	.04061
15.119	34.595	7.7964	.45325	.47526	.02200	.08546
15.404	34.997	7.2935	.82000	.91571	.09562	.16527
15.582	35.387	6.8063	1.265	1.574	.3092	.2732
15.741	35.657	6.0778	1.573	2.285	.7128	.38045
15.884	35.839	5.1589	1.591	2.752	1.1613	.4533
16.016	36.194	4.2811	1.1469	2.8203	1.4033	.4865
16.139	36.445	3.5951	1.2139	2.5827	1.3688	.4963

Figure 6. Equation of State p = p(ε) for the Neutron Star

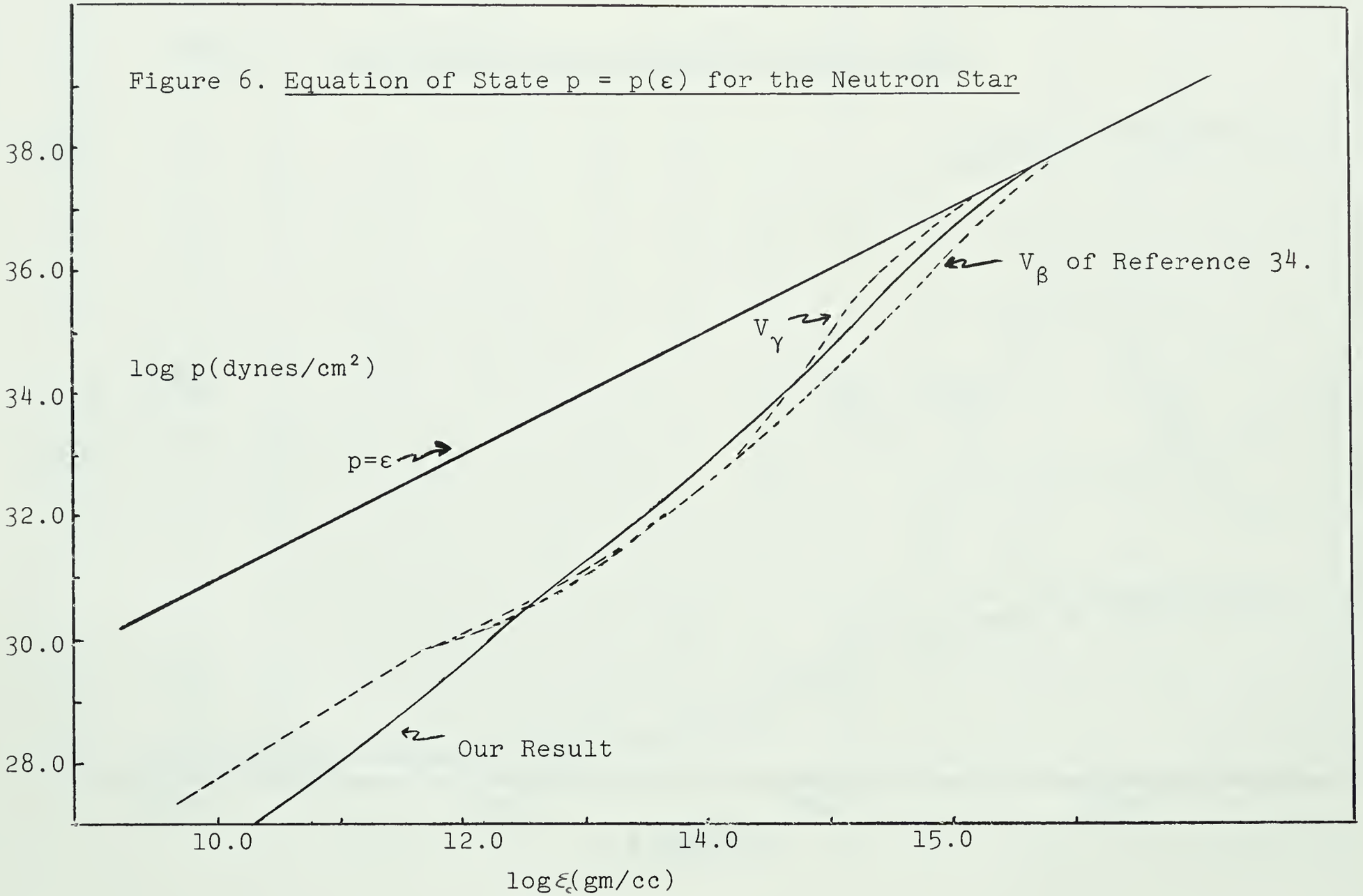
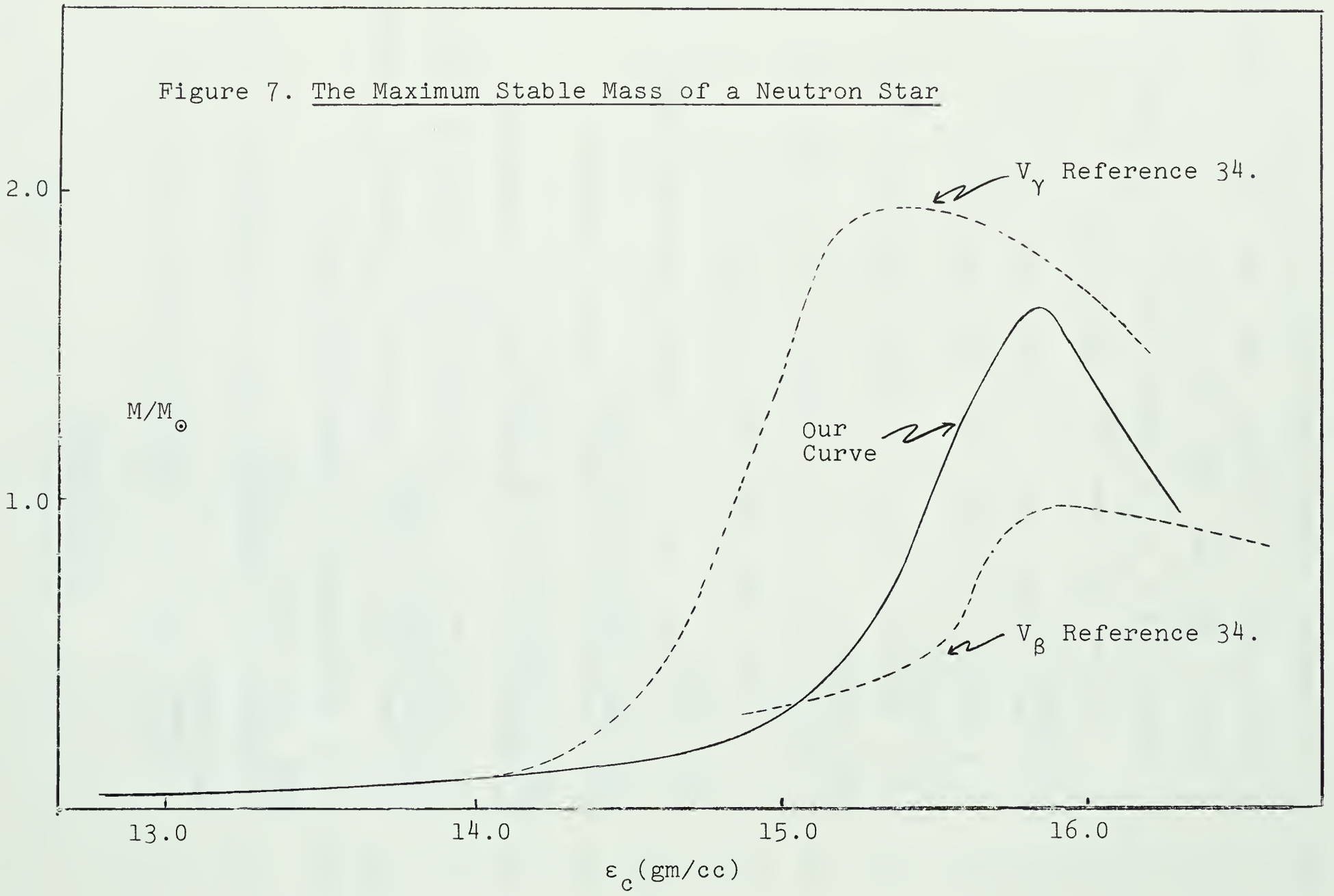


Figure 7. The Maximum Stable Mass of a Neutron Star



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